MST 204.25

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods.

mathematical models and methods

Unit 25
Functions of more than one variable

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Unit 25
Functions of more than one variable

Prepared for the Course Team
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The Open University

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First published 1982. Reprinted 1983, 1985, 1989, 1992, 1994, 1996.

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ISBN 0 335 14054 8

Printed and bound in the United Kingdom by Staples Printers Rochester Limited, Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT.

This text forms part of the correspondence element of an Open University Second Level Course.

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Introduction

So far in this course a large part of the mathematical modelling has been concerned with how various physical quantities (such as the positions of particles) vary with *time*. This unit is the first in a group of three introducing ideas that are useful for modelling physical quantities that depend on position in *space* instead of, or as well as, *time*. The main example of space dependence we have seen so far was in the unit on *heat transfer* where, for example, we were able to calculate how the temperature inside the wall of a house depends on distance from the inside face of the wall. Matters were fairly simple there because we only dealt with situations where we could find a Cartesian or polar coordinate system such that the temperature was a function of just *one* of the coordinates. But this simplification is not always possible, and so we need a way of dealing with situations where the temperature depends on all three space coordinates and on time.

To describe this type of dependence we introduce in this unit the idea of a *function of several variables*. In particular, we will be concerned with how we can extend the idea of *differentiation* to such functions. The extension will be applied in this unit to the mathematical problem of investigating maxima and minima of functions of two variables, and in the next unit to physical problems such as generalizing the formulation of Fourier's law of heat conduction to general temperature distributions.

We will explain, wherever we can, the geometric interpretation of what we are doing analytically. We will also make some effort to show how our new ideas and techniques can be used to calculate approximate values for quantities which it would be nasty or impossible to determine accurately; so you should have your calculator handy, particularly throughout your study of Section 2 of the unit.

Study guide

Before watching the television programme you should at least read quickly through Section 1 and through Subsection 4.1 up to and including Exercise 1 of that section. Otherwise the sections should be read in their natural order. The two audio tape sessions deal principally with the solving of simultaneous equations which often arise in finding maxima and minima of functions of two variables. The first tape session follows the television programme and you should watch the programme and then listen to this tape during the same study session.

This unit contains many exercises—more than you could reasonably do during your study of the unit. Concentrate, therefore, upon attempting those exercises dealing with the techniques you find most difficult, leaving the remaining exercises for revision later.

1 First-order partial derivatives

1.1 Functions of more than one variable

In real life the quantities in which we are interested often depend on several variables rather than on just one. Let us look at some examples of this assertion.

The height of a hill at map co-ordinates (x, y)

Consider a map of a square region of countryside containing a hill (see Figure 1). In order to be able to refer to the position of any point P on this region of the Earth's surface we may agree to draw two fixed co-ordinate axes on the map at right angles to each other and then specify the location (x, y) of the point representing P on the map by giving its distance from each axis in turn.

Maps generally show contour lines (that is, lines of constant height above sea-level). Using them we can specify the location of P more precisely by assigning to

each point (x, y) on the map a third number h, say, where h is the height of P above sea level.

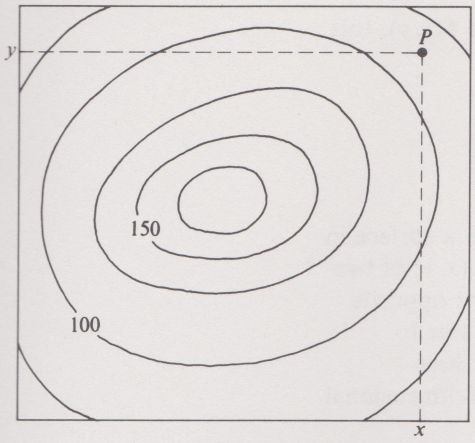


Figure 1. Part of a map

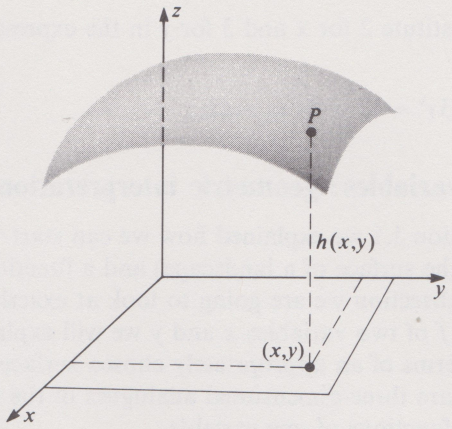


Figure 2. The corresponding region of countryside

Corresponding to each choice of a pair of co-ordinates (x, y) there is a corresponding number h. So we say that h is a function of (that is, is completely determined by) the *two* variables x and y; we often write $h(x, y)$ instead of h to remind ourselves of this (see Figure 2).

The temperature distribution over a dinner plate at some given time

Suppose we consider a (flat) circular dinner plate, and agree to describe points on its surface by means of a rectangular (Cartesian) co-ordinate system using the centre of the plate as origin (see Figure 3). The temperatures at different points on the plate may not be the same. For example the temperature might be higher at Q than at P. Nevertheless, at any given moment each point (x, y) on the plate has a well-defined temperature T (in degrees Celsius). We can write this as $T(x, y)$ to remind us that the value of T depends on the particular choice we make of x and y. We say that T is **a function of** the **two variables** x and y; compare this with functions of a single variable, such as $\sin x$, which you have been using so far. We call T the **dependent variable** and x and y the **independent variables**.

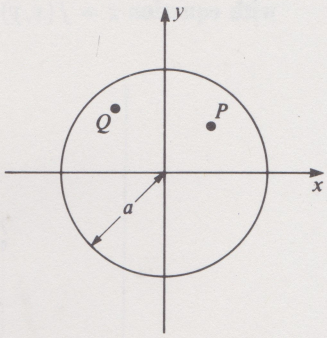


Figure 3. A dinner plate

Notice the natural restriction on the variables x and y arising from the situation that we are describing: $T(x, y)$ has only been defined for points (x, y) lying on the plate. Putting this another way, the set of allowable values of x and y are those satisfying $x^2 + y^2 \leq a^2$ where a is the radius of the plate. This set is called the **domain** of the function $T(x, y)$.

It is often possible in mathematical modelling to express functions in terms of convenient formulae. For example, we might be able to represent $T(x, y)$ by the following formula:

$$T(x, y) = 100e^{-(x^2+y^2)/a^2} \qquad (x^2 + y^2 \leq a^2).$$

This formula corresponds to a plate which is hottest at its centre (where $T(0, 0) = 100$) and coolest at its rim (for example, $T(0, a) = 100e^{-1} = 36.787\ldots$).

The temperature distribution over a dinner plate during a meal

In this case the temperature T at a given point (x, y) on the plate in Figure 3 will depend not only on the co-ordinates x and y of the point which we are considering but also on time t; that is, on the particular instant which we specify during the meal. Since the value of T now depends on the three quantities x, y and t, we say that T is **a function of** the **three variables** x, y and t, and express this dependence by writing T as $T(x, y, t)$.

The following example shows how to evaluate a function of two variables.

Example 1

If $f(x, y) = 3x^2 - 2y^3$, what is the value of $f(2, 3)$?

Solution

To get this quantity we substitute 2 for x and 3 for y in the expression $f(x, y)$; this gives

$$f(2, 3) = 3(2)^2 - 2(3)^3 = 12 - 54 = -42.$$

1.2 Functions of two variables: geometric interpretation

At the beginning of Subsection 1.1 we explained how we can start with a surface in three dimensions (actually the surface of a landscape) and a function $h(x, y)$ of two variables x and y. In this subsection we are going to look at exactly the opposite approach: given a function f of two variables x and y we will explain how to 'represent' the function in terms of an appropriately chosen surface in three dimensions. These surfaces are three-dimensional analogues of the two-dimensional graphs we use to represent functions of one variable.

Corresponding to each point (x, y) in the domain of $f(x, y)$ there is a unique real number z given by the formula $z = f(x, y)$. We may interpret this geometrically in the following way.

Corresponding to each point Q with coordinates $(x, y, 0)$ in the x, y-plane for which (x, y) belongs to the domain of f, we can construct a perpendicular upwards from Q of height $z = f(x, y)$; if we let P be the point (x, y, z) (see Figure 4) we say that the point P represents the value of the function $f(x, y)$ at the point (x, y). If we look at the set of all such points P that correspond to points (x, y) in the domain of $f(x, y)$, we have a set of points in three dimensions; we call this set the **surface with equation** $z = f(x, y)$ (see Figure 5).

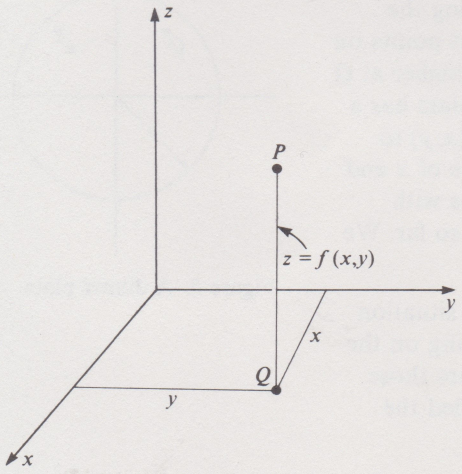


Figure 4

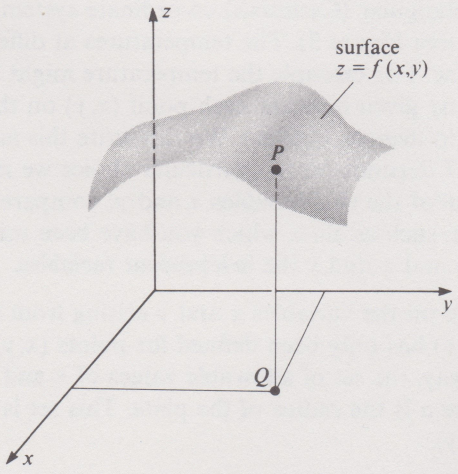


Figure 5

For example, the function

$$f(x, y) = -\tfrac{2}{3}x - 2y + 2$$

can be associated with the surface whose equation is $z = -\frac{2}{3}x - 2y + 2$. Since this equation can be written in the form $\frac{2}{3}x + 2y + z = 2$, the surface is a plane. Noticing that the points $(3, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$ lie on the plane, we see that the plane must be as illustrated in Figure 6.

Notice that the surface with equation

$$z = x^2 + y^2 + 1$$

(which is the surface corresponding to the function $f(x, y) = x^2 + y^2 + 1$) can be shown to be the surface that we can generate by taking the parabola $z = x^2 + 1$ in the $y = 0$ plane and rotating it about the z-axis (see Figure 7).

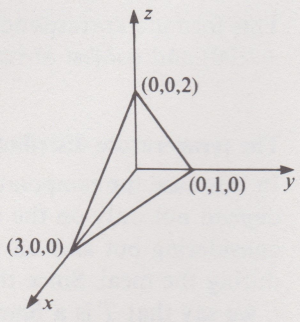


Figure 6. The surface $z = -\frac{2}{3}x - 2y + 2$

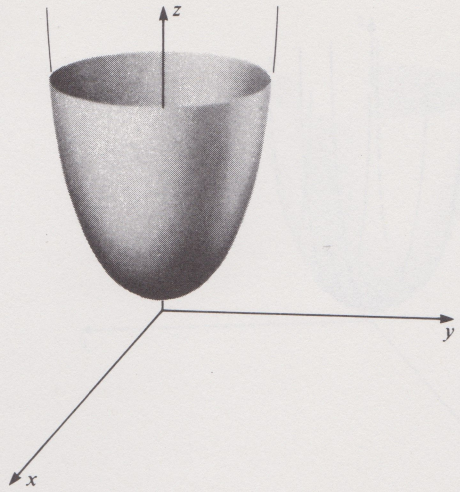


Figure 7. The surface $z = x^2 + y^2 + 1$

In Subsection 1.3 we are going to study **section functions**. These are constructed from functions of several variables by holding all the variables except one fixed (i.e. constant). For example, in the function considered above which is

$$f(x, y) = x^2 + y^2 + 1$$

we may hold y at the fixed value 2; this defines a new function of the *single* variable x which we may denote by

$$f(x, 2) = x^2 + 2^2 + 1 = x^2 + 5.$$

This function, $x^2 + 5$, is the section function of $f(x, y)$ corresponding to fixing y at the value 2. As another example, we can hold x instead of y fixed in the function $x^2 + y^2 + 1$: suppose we fix x to be 3; then the result is a function of the *single* variable y, namely the function

$$f(3, y) = 3^2 + y^2 + 1 = y^2 + 10.$$

This function, $y^2 + 10$, is the section function of $f(x, y)$ with x fixed at 3.

The graph of any section function can be obtained by taking the intersection of the surface $z = f(x, y)$ with a plane $x =$ constant or $y =$ constant. For example, the section function $f(x, 2)$ can be represented by the curve $z = f(x, 2)$ in the plane $y = 2$ (which is parallel to the $y = 0$ coordinate plane). This is illustrated in Figure 8 on the next page, for the case $f(x, y) = x^2 + y^2 + 1$, if we take $b = 2$ there. In general, the curves representing section functions are obtained by looking at slices through the surfaces in planes parallel to the $x = 0$ or $y = 0$ planes.

To get a feeling for what section functions are, let us see what is the shape of the intersection of the surface $z = x^2 + y^2 + 1$ $(= f(x, y))$, as shown in Figure 8, with the plane $y = b$ for any fixed value of b. (That is, we will find the shape of the graph of the section function $f(x, b)$.)

This intersection is the set of points (x, y, z) that satisfy the equations

$$y = b \quad \text{and} \quad z = x^2 + y^2 + 1;$$

that is,

$$y = b \quad \text{and} \quad z = x^2 + (b^2 + 1).$$

This last equation represents a parabola whose axis has equation $x = 0$, $y = b$ and whose vertex is the point $(0, b, b^2 + 1)$.

When $b = 0$, the vertex has co-ordinates $(0, 0, 1)$; and, as $|b|$ increases (in other words, as the plane $y = b$ moves away from the x,z-coordinate plane) the vertex rises (the sign of b makes no difference). By putting together the curves 'given by' all the different section functions, we can construct the whole surface as illustrated in Figure 9.

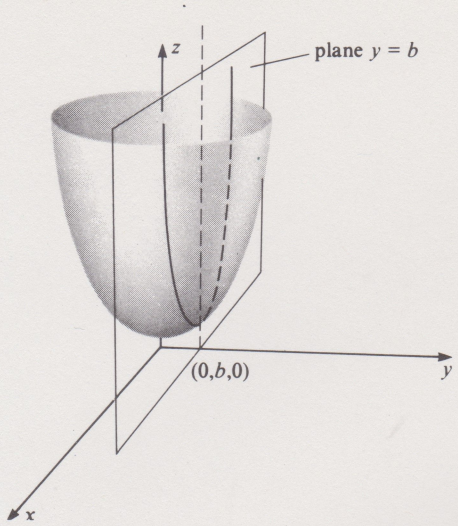


Figure 8. Intersection of surface with plane $y = b$

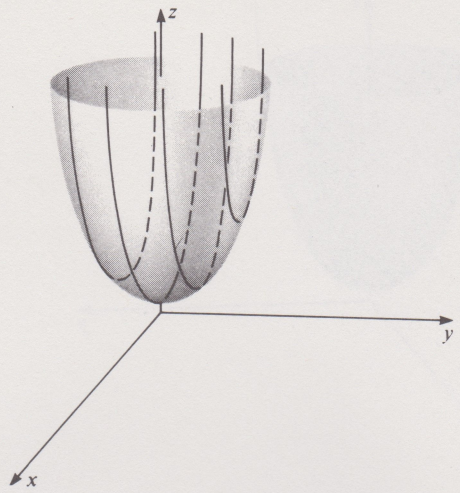


Figure 9. Intersection of surface with planes $y =$ constant

1.3 First-order partial derivatives

In Section 2 we will be interested in the behaviour of functions of several variables near specific points; that is, when all the independent variables stay close to specified values. But to begin with, instead of allowing all the quantities to vary, let us fix all of them except one and allow just that one to vary. For example, suppose that we are interested in the behaviour of a function $f(x, y)$ when x and y are near some given values a and b. The function $f(x, b)$ is a section function of f, whose value at $x = a$ is $f(a, b)$. As x varies, so does the value of the function $f(x, b)$; so we can find the derivative of this function with respect to x at the point $x = a$.

In exactly the same way we can define the derivative of the section function $f(a, y)$ with respect to y at the point $y = b$. As we will see in the next example, this derivative does not necessarily have the same value as the derivative in the previous paragraph; so we can really call neither *the* derivative of f at (a, b). Instead, we call the derivative of $f(x, b)$ with respect to x at $x = a$ **the partial derivative of f with respect to x at (a, b)**; symbolically we write it as

$$\frac{\partial f}{\partial x}(a, b);$$

that is, 'partial df by dx'.

Similarly the derivative of $f(a, y)$ with respect to y at $y = b$ we will call **the partial derivative of f with respect to y at (a, b)**, and we will denote it by

$$\frac{\partial f}{\partial y}(a, b);$$

that is, 'partial df by dy'. Since only one differentiation is involved, these partial derivatives are said to be of **first order**.

Warning: never use a 'd' when you mean a '∂'; functions of several variables do not have derivatives, but they do have partial derivatives.

Example 2

Determine both partial derivatives of the function

$$f(x, y) = x^2 - y^3 + 2xy^2$$

at the point $(2, -1)$.

Solution

To calculate the partial derivative $\dfrac{\partial f}{\partial x}(2, -1)$, we fix y to have the value -1, and look at the resulting section function

$$f(x, -1) = x^2 + 1 + 2x.$$

To find its derivative at $x = 2$, we first differentiate in the usual way, to get

$$\frac{d}{dx}f(x, -1) = \frac{d}{dx}(x^2 + 1 + 2x) = 2x + 2.$$

Setting $x = 2$ gives the derivative which we have defined to be the partial derivative (with respect to x) of the original function of two variables at $(2, -1)$:

$$\frac{\partial f}{\partial x}(2, -1) = 2 \times 2 + 2 = 6.$$

Similarly, to calculate $\frac{\partial f}{\partial y}(2, -1)$ we first set $x = 2$ and look at the section function

$$f(2, y) = 4 - y^3 + 4y^2.$$

To find its derivative at $y = -1$ we differentiate with respect to y, obtaining

$$\frac{d}{dy}f(2, y) = \frac{d}{dy}(4 - y^3 + 4y^2) = -3y^2 + 8y.$$

Then we set $y = -1$, obtaining by the definition of $\frac{\partial f}{\partial y}$,

$$\frac{\partial f}{\partial y}(2, -1) = -3(-1)^2 + 8(-1) = -11.$$

Notice that the two partial derivatives are not equal, so that it really does make sense to distinguish between them. Geometrically $\frac{\partial f}{\partial x}(2, -1)$ is the slope of the tangent line to the surface $z = f(x, y)$ which lies in the plane $y = -1$, and $\frac{\partial f}{\partial y}(2, -1)$ is the slope of the tangent line to the surface which lies in the plane $x = 2$; obviously in general these slopes are unequal.

We can calculate the partial derivatives of $f(x, y)$ at an arbitrary point (a, b) in just the same way as for $(2, -1)$. To calculate $\frac{\partial f}{\partial x}(a, b)$ we first fix y at the value b and consider the section function

$$f(x, b) = x^2 - b^3 + 2xb^2.$$

Differentiation gives

$$\frac{d}{dx}f(x, b) = 2x + 2b^2$$

and setting $x = a$ gives the partial derivative

$$\frac{\partial f}{\partial x}(a, b) = 2a + 2b^2.$$

Question

Find $\frac{\partial f}{\partial y}(a, b)$ in this case.

Answer

The section function to be differentiated is

$$f(a, y) = a^2 - y^3 + 2ay^2.$$

Differentiating with respect to y gives

$$\frac{d}{dy}f(a, y) = -3y^2 + 4ay$$

and so

$$\frac{\partial f}{\partial y}(a, b) = -3b^2 + 4ab.$$

Because a and b are quite general we could just as well have taken x and y in place of a and b. This defines the function $\frac{\partial f}{\partial x}(x, y)$, abbreviated to $\frac{\partial f}{\partial x}$, as a function of x and y. To calculate $\frac{\partial f}{\partial x}$ we just calculate $\frac{\partial f}{\partial x}(a, b)$ and replace a by x and b by y in the resulting expression. Once you get used to this process it saves time to calculate partial derivatives at (x, y) directly instead of at (a, b) first. For example, given the function

$$f(x, y) = x^2 - y^3 + 2xy^2$$

we calculate $\frac{\partial f}{\partial x}$ by differentiating with respect to x, *while treating y as a constant.* This gives

$$\frac{\partial f}{\partial x}(x, y) = 2x + 0 + 2y^2.$$

This calculation is called **differentiating $f(x, y)$ partially with respect to x.**

Question

Find $\frac{\partial f}{\partial y}$ in this case.

Answer

We differentiate $f(x, y)$ with respect to y, treating x as a constant. This gives

$$\frac{\partial f}{\partial y} = 0 - 3y^2 + 4xy.$$

You should check that these formulae for $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ agree with those for $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ found earlier.

Example 3

Determine the first-order partial derivatives at (x, y) of the following functions:

(i) $g(x, y) = \sin(3x - 2y)$;

(ii) $h(x, t) = e^{4x+5t} \sin(2x + 3t)$.

Solution

(i) Just as in the previous remarks, differentiating g partially with respect to x and y in turn, we get

$$\frac{\partial g}{\partial x}(x, y) = 3\cos(3x - 2y) \quad \text{and}$$

$$\frac{\partial g}{\partial y}(x, y) = -2\cos(3x - 2y).$$

(ii) Differentiating h partially with respect to x and t in turn, we get

$$\frac{\partial h}{\partial x}(x, t) = 4e^{4x+5t} \sin(2x + 3t) + 2e^{4x+5t} \cos(2x + 3t), \quad \text{and}$$

$$\frac{\partial h}{\partial t}(x, t) = 5e^{4x+5t} \sin(2x + 3t) + 3e^{4x+5t} \cos(2x + 3t).$$

Although there is no convenient geometric interpretation, we can define first-order partial derivatives of functions of more than two variables in a similar way; keep all the variables but one fixed for the moment, and differentiate in the usual way with respect to the remaining variable.

Example 4

Determine the first-order partial derivatives of the function

$$f(x, y, t) = x^2y^3t^4 + 2xy + 4t^2x^2 + y.$$

Solution

Differentiating partially with respect to x, y and t in turn, we get

$$\frac{\partial f}{\partial x}(x, y, t) = 2xy^3t^4 + 2y + 8t^2x;$$

$$\frac{\partial f}{\partial y}(x, y, t) = 3x^2y^2t^4 + 2x + 1;$$

$$\frac{\partial f}{\partial t}(x, y, t) = 4x^2y^3t^3 + 8tx^2.$$

Exercise 1

Determine the first-order partial derivatives of the following functions:

(i) $f(x, y) = x^2 \log y \qquad (y > 0)$;

(ii) $g(x, y) = 1 + x - y^2 + 2x^2y$;

(iii) $h(x, y, t) = xe^t - y^2e^{2t}$.

What are the values of $\frac{\partial f}{\partial y}(1, 1)$ and $\frac{\partial h}{\partial t}(4, 1, 0)$?

[*Solution on p. 39*]

Summary of Section 1

1. A **function $f(x, y)$ of two variables** x and y is one whose value is determined by the values of x and y. We call x and y the **independent variables**; a variable equal to $f(x, y)$ is called the **dependent variable**.
2. If $f(x, y)$ is a function of two variables x and y, the **surface** $z = f(x, y)$ in three dimensions *represents* the function $f(x, y)$.
3. The **section functions** of $f(x, y)$ at (a, b) are $f(x, b)$, regarded as a function of x with b regarded as a constant, and $f(a, y)$, regarded as a function of y with a constant.
4. The **first-order partial derivatives of $f(x, y)$ at (a, b) with respect to x and y**, written $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$, are the derivatives at a and b of the section functions $f(x, b)$ and $f(a, y)$ respectively.
5. The **first-order partial derivatives of $f(x, y)$ with respect to x and y** are written $\frac{\partial f}{\partial x}\left(\text{or } \frac{\partial f}{\partial x}(x, y)\right)$ and $\frac{\partial f}{\partial y}\left(\text{or } \frac{\partial f}{\partial y}(x, y)\right)$ and are the functions of two variables obtained by replacing a by x and b by y in $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ respectively.
6. Similarly we can define functions of more than two variables (for example, $f(x, y, t)$), their section functions and their first-order partial derivatives.

End of section exercises

Exercise 2

Determine the first-order partial derivatives of the following functions:

(i) $f(x, y) = 3 + x - x^3 - 2xy + y^2$;

(ii) $g(x, y) = x \sin(x - y)$;

(iii) $h(x, y, t) = e^{x-t} \cos(y + t)$.

What are the values of $\frac{\partial f}{\partial x}(1, -1)$ and $\frac{\partial h}{\partial t}\left(0, \frac{\pi}{2}, 0\right)$?

[*Solution on p. 39*]

Exercise 3

Determine the first-order partial derivatives of the following functions:

(i) $T(x, y) = x^2y - 2xy^4$;

(ii) $f(x,y) = e^{-xy}\sin x + x^2 - y^2$;

(iii) $g(x,t) = e^{-t/x}\sin(2t)$;

(iv) $h(x,y,z) = xz^2\sin(x/y)$;

(v) $k(r,\theta) = \dfrac{1}{1 - 2r\cos\theta + r^2}$.

[*Solution on p. 39*]

2 Some applications of partial derivatives

2.1 Preliminaries

The partial derivatives that we considered in Section 1 gave us an idea of the behaviour of a function $f(x,y)$ near a particular point when only one of the variables is allowed to change. In this section we meet Taylor polynomials of first order for $f(x,y)$ that describe the approximate behaviour of $f(x,y)$ near a given point. These are tied in with the equation of the tangent plane at a point on the surface $z = f(x,y)$ (giving an interesting connection between analytic and geometric ideas) and are used to approximate to the values of functions and to estimate small errors due to uncertainties in experimental data. The key idea that runs through the section is that, near a point P on a surface, the tangent plane at P to the surface is more-or-less the same as the surface itself.

Here is a summary of facts about Taylor polynomials that we will be using in the rest of this unit and generalizing to functions of two variables; you may have seen them stated previously in slightly different ways. Suppose that g is a function of a single variable which is 'well behaved' for x near a in the sense that the graph $y = g(x)$ near $x = a$ is smooth.

The function

$$q_1(x) = g(a) + g'(a)(x - a) \qquad (1)$$

is the **first-order Taylor polynomial for g at a**, and

$$g(x) \simeq q_1(x) = g(a) + g'(a)(x - a) \qquad (2)$$

when x is near a.

The symbol '$\simeq$' means 'is approximately equal to'.

The function

$$q_2(x) = g(a) + g'(a)(x - a) + \tfrac{1}{2}g''(a)(x - a)^2 \qquad (3)$$

is the **second-order Taylor polynomial for g at a**, and again

$$g(x) \simeq q_2(x) = g(a) + g'(a)(x - a) + \tfrac{1}{2}g''(a)(x - a)^2 \qquad (4)$$

when x is near a.

The error in the approximation $g(x) \simeq q_1(x)$ is small in comparison with $(x - a)$ when $(x - a)$ is small. The error in the approximation $g(x) \simeq q_2(x)$ is small in comparison with $(x - a)^2$ when $(x - a)$ is small. The proof of these statements about this error is beyond the scope of this course.

2.2 Taylor polynomials of first order

Functions of two variables are very often specified by formulae in such a way that we can determine their values at specific points by simple substitution. For example, if $f(x,t) = x^4 - 3x^3t + t^3$, then

$$f(0.3, -0.1) = (0.3)^4 - 3(0.3)^3(-0.1) + (-0.1)^3 = 0.0152.$$

Sometimes however it is more important to have a convenient approximation to the values taken by a function (over some range of values for its independent variables) than to go through the labour of calculating it exactly every time it is needed. The Taylor approximation enables us to do this for functions which are

well-behaved; that is, ones corresponding to surfaces $z = f(x, y)$ that are **smooth** in the sense that they have no folds or breaks (see Figure 1). Practically all the functions we come across are well-behaved in this sense.

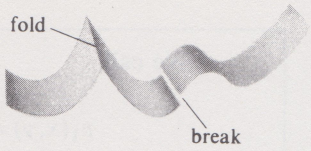


Figure 1

So let us consider a function $f(x, y)$ of two independent variables x and y near some fixed point (a, b). Obviously for x and y near to a and b the value of $f(x, y)$ is fairly near to the value of $f(a, b)$. But let us look for a better approximation to $f(x, y)$ having a form similar to (2) but generalized to two variables:

$$f(x, y) \simeq f(a, b) + A(x - a) + B(y - b)$$

where A and B are constants to be found later; by this approximation we mean that the difference

$$f(x, y) - [f(a, b) + A(x - a) + B(y - b)]$$

is small in comparison with $(x - a)$ and $(y - b)$ when both $(x - a)$ and $(y - b)$ are small. In other words, we wish to take into account the fact that $f(x, y)$ is not necessarily constant near to (a, b) by including multiples of $(x - a)$ and $(y - b)$ in the estimate; these extra terms, since they both vanish when $x = a$ and $y = b$ and are first-order powers of $(x - a)$ and $(y - b)$, are called **first-order correction terms** to the original crude estimate $f(x, y) \simeq f(a, b)$. In the calculation below to determine the values of A and B, we will at each stage retain terms which will contribute to either A or B but discard those which would only lead in the long run to terms of a smaller order of magnitude than $(x - a)$ and $(y - b)$.

So let us apply Taylor's approximation (2) to the function $f(x, y)$, where for the moment y is kept fixed and x is allowed to vary: this will enable us to relate $f(x, y)$ to $f(a, y)$, and afterwards we can relate $f(a, y)$ to $f(a, b)$ by applying Taylor's approximation again, this time with x held fixed and y allowed to vary. For the first step, then, we apply Taylor's first-order approximation (2) with $g(x)$ replaced by the function $f(x, y)$ where x is the variable and y is treated as a constant. Then $g(x)$ in (2) is replaced by $f(x, y)$, and the term $g(a)$ is replaced by $f(a, y)$; in the last term the derivative $g'(a)$, which means $\dfrac{dg}{dx}(a)$, must be replaced by $\dfrac{\partial f}{\partial x}(a, y)$ to show that we are treating y as a constant. So equation (2) becomes

$$f(x, y) \simeq f(a, y) + \frac{\partial f}{\partial x}(a, y)(x - a) \qquad (x - a \text{ small}). \tag{5}$$

Now let us apply Taylor's approximation (2) to the section function $f(a, y)$ that arose in (5); here a is fixed, and y is the variable. Then the approximation (2) becomes

$$f(a, y) \simeq f(a, b) + \frac{\partial f}{\partial y}(a, b)(y - b) \qquad (y - b \text{ small}). \tag{6}$$

We also need an approximation to the partial derivative $\dfrac{\partial f}{\partial x}(a, y)$ in (5). It is sufficient to use the crude approximation

$$\frac{\partial f}{\partial x}(a, y) \simeq \frac{\partial f}{\partial x}(a, b) \qquad (y - b \text{ small}) \tag{7}$$

for if we used any better approximation the additional terms which would appear on the right-hand side of (7) would, after multiplication by the factor $(y - b)$ in (6), simply give us terms in our final approximation which are too small to matter here.

Putting the estimates (6) and (7) into the right-hand side of (5) we get the following result, which holds when $(x - a)$ and $(y - b)$ are small:

$$f(x, y) \simeq f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b). \tag{8}$$

The fact that the right-hand side of (8) is indeed a formula in x and y of the desired type leads us to the following definition and result.

We call

$$p_1(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \qquad (9)$$

the Taylor polynomial of first order for f at (a,b). Then, for x and y near to a and b, $f(x,y)$ is approximately equal to $p_1(x,y)$.

Notice that p_1 depends upon the point (a,b). A different point will give a different polynomial even though the notation p_1 does not reflect this fact (see Exercise 1).

The error is small in comparison with $(x-a)$ and $(y-b)$ when both $(x-a)$ and $(y-b)$ are small. Quite often the approximation to f given by p_1 will be good enough for our purposes, and easier to calculate than f itself.

Example 1

Determine the Taylor polynomial of first order for the function $f(x,y) = \sqrt{x\ y}$ at the point $(1,1)$.

Solution

Differentiating with respect to x and y in turn, we get

$$\frac{\partial f}{\partial x} = \frac{1}{2}\sqrt{y/x} \text{ and } \frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{x/y}.$$

It follows that

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{2}, \qquad \frac{\partial f}{\partial y}(1,1) = \frac{1}{2} \qquad \text{and} \qquad f(1,1) = 1.$$

Hence, by (9), the Taylor polynomial of first order for f at $(1,1)$ is

$$\begin{aligned} p_1(x,y) &= f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \\ &= 1 + \tfrac{1}{2}(x-1) + \tfrac{1}{2}(y-1). \end{aligned}$$

We could multiply out the brackets and simplify this to $\frac{1}{2}x + \frac{1}{2}y$, but it usually saves work to leave the Taylor polynomial in the form shown here.

Exercise 1

Determine the Taylor polynomial of first order for each of the following functions at the indicated point:

(i) $f(x,y) = \sqrt{x\ y}$ at the point $(4,9)$;

(ii) $f(x,y) = x^5 + 3x^2y - xy^3$ at the point $(1,2)$.

[*Solution on p. 39*]

The following example shows how Taylor polynomials can often be used to approximately evaluate expressions without using a calculator.

Example 2

Using an appropriate Taylor polynomial estimate the quantity

$$(7.91 \times 124.2)^{1/3}.$$

Solution

Obviously a crude estimate is $(8 \times 125)^{1/3} = 10$. Let us see whether, using a suitably chosen function f and its first-order Taylor polynomial at a suitably chosen point, we can improve this estimate. Let us consider the function

$$f(x,y) = x^{1/3}y^{1/3}$$

in a neighbourhood of the point $(8, 125)$. (There are many other possible choices of f and (a,b), but these are the obvious ones to try.) Then,

$$\frac{\partial f}{\partial x} = \tfrac{1}{3}x^{-2/3}y^{1/3} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \tfrac{1}{3}x^{1/3}y^{-2/3};$$

so that $f(8,125) = 10$, $\dfrac{\partial f}{\partial x}(8,125) = \tfrac{1}{3}(8)^{-2/3}(125)^{1/3} = \tfrac{5}{12}$, and

$$\frac{\partial f}{\partial y}(8,125) = \tfrac{1}{3}(8)^{1/3}(125)^{-2/3} = \tfrac{2}{75}.$$

The Taylor polynomial of first order for f at $(8, 125)$ is hence

$$p_1(x, y) = 10 + \tfrac{5}{12}(x - 8) + \tfrac{2}{75}(y - 125).$$

(As before we do not multiply out the brackets.) The quantity which we are trying to estimate is given by $f(7.91, 124.2)$. An approximation to it is therefore

$$p_1(7.91, 124.2) = 10 + \tfrac{5}{12}(-0.09) + \tfrac{2}{75}(-0.8) = 9.941166\ldots \quad .$$

Since the value of $f(7.91, 124.2)$ *is* $9.941059\ldots$ (from your calculator) our answer, while only an estimate, is certainly a lot better than the original crude estimate, and did not involve the use of a calculator.

Exercise 2

Using an appropriate Taylor polynomial estimate the value of

$$\sqrt{(2.98)^2 + (4.01)^2}.$$

[*Solution on p. 40*]

Exercise 3

The geometric mean m of two numbers x and y is given by the formula

$$m = m(x, y) = \sqrt{xy}.$$

(For example, if $x = 4$ and $y = 9$, their geometric mean is 6.) Using an appropriate Taylor polynomial, estimate the geometric mean of 3.96 and 8.91.

[*Solution on p. 40*]

In calculations involving Taylor polynomials we are often more concerned with the difference between two values of a function than with the values themselves. To this end we will introduce some new notation for rewriting Taylor's approximation. For x and y near a and b

$$f(x, y) \simeq f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Let us write δx in place of $x - a$, δy in place of $y - b$, and δf in place of $f(x, y) - f(a, b)$; then the last formula can be rewritten in the form

$$\delta f \simeq \frac{\partial f}{\partial x}(a, b)\delta x + \frac{\partial f}{\partial y}(a, b)\delta y \qquad (\delta x, \delta y \text{ small}). \tag{10}$$

In other words, if the values of x and y are changed by small amounts δx and δy, the consequent change in f, which we are calling δf, is approximately equal to the right-hand side of (10).

In our next example we will use Taylor's approximation in the convenient form (10). We will also use the following result whose proof is obvious.

For any real numbers a and b,

$$|a + b| \leq |a| + |b|.$$

This is called the **triangle inequality**.

Example 3

The focal length f of a convex lens can be determined from measurement with the help of the formula

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}, \tag{11}$$

where p and q are the distances from the lens of an object and of its image respectively. If p and q are measured to be approximately 4.6 m and 0.4 m with maximum error of 0.1 m and 0.01 m respectively, find an estimate for the focal length of the lens. Also, estimate the maximum error in this approximation.

Solution

Using the measured values for p and q, the approximate value, f, of the focal length of the lens is given by

$$\frac{1}{f} = \frac{1}{4.6} + \frac{1}{0.4},$$

that is, $f = 0.368\,m$.

Let us now use our approximation formula (10) to determine the likely accuracy of this estimate for f. If we put $p = 4.6$ and $q = 0.4$, we know that the actual values of the distances that we have approximated by these values of p and q are given by $p + \delta p$ and $q + \delta q$, where

$$|\delta p| \leq 0.1 \qquad \text{and} \qquad |\delta q| \leq 0.01. \tag{12}$$

If we let f denote the value given by the formula (11) corresponding to the values $p = 4.6$ and $q = 0.4$, and we denote the actual focal length of the lens by $f + \delta f$, then (10) tells us that

$$\delta f \simeq \frac{\partial f}{\partial p}(4.6, 0.4)\delta p + \frac{\partial f}{\partial q}(4.6, 0.4)\delta q. \tag{13}$$

But (11) tells us that $f = \dfrac{pq}{p+q}$, so that in general

$$\frac{\partial f}{\partial p} = \frac{q^2}{(p+q)^2} \quad \text{and} \quad \frac{\partial f}{\partial q} = \frac{p^2}{(p+q)^2}.$$

Hence,

$$\frac{\partial f}{\partial p}(4.6, 0.4) = 0.0064 \quad \text{and} \quad \frac{\partial f}{\partial q}(4.6, 0.4) = 0.8464.$$

Substituting these into (13) we deduce that

$$\delta f \simeq 0.0064\delta p + 0.8464\delta q.$$

Taking the modulus of both sides, gives

$$\begin{aligned} |\delta f| &\simeq |0.0064\delta p + 0.8464\delta q| \\ &\leq 0.0064|\delta p| + 0.8464|\delta q| \qquad \text{(triangle inequality)} \\ &\leq 0.0064 \times 0.1 + 0.8464 \times 0.01 \qquad \text{(from (12))} \\ &= 0.009104\,\text{m}. \end{aligned}$$

We conclude that the focal length of the lens is approximately 0.368 m, with a maximum error of about 0.009 m.

Exercise 4

In surveying a triangular plot of land two of its sides were measured and found to be 160 m and 210 m, with an error of at most 0.1 m in either measurement; the included angle was $\pi/3$ exactly. Estimate the maximum error in calculating the length of the third side from this data using the formula $c = \sqrt{a^2 + b^2 - 2ab\cos C}$ (see Figure 2).

[*Solution on p. 40*]

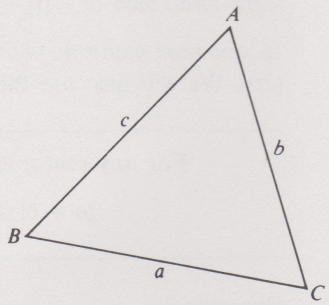


Figure 2

Finally we remark that we can extend the approximation formulae (8) and (10) to functions of more than two variables in a completely analogous way. For example, if f is a function of the independent variables x, y and t, the analogous formula for (10) is:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta t.$$

Exercise 5

The length, L, of the diagonal of a rectangular tank of length x, breadth y, and height z, is given by the formula $L = \sqrt{x^2 + y^2 + z^2}$.

If x, y and z are measured to be 1 m, 2 m and 2 m respectively, but these measurements are each accurate only to 0.03 m, estimate the maximum resulting error in the length of the diagonal.

[*Solution on p. 40*]

2.3 Tangent planes

When you studied the calculus of functions of one variable, an important role was played by the tangent lines to the curve $y = g(x)$ at points $x = a$; these had equations of the form

$$y = mx + c, \qquad \text{where } m = g'(a).$$

We have already seen (in Subsection 1.2) that, with functions $f(x, y)$ of two variables, we can associate surfaces $z = f(x, y)$ in three dimensions. The analogues of lines in two dimensions are planes in three dimensions, and the analogues of tangent lines to curves $y = g(x)$ are tangent planes to surfaces. So what is the equation of the tangent plane at some arbitrary point $(a, b, f(a, b))$ on a given surface $z = f(x, y)$? We saw in the last subsection that if x and y are near to some given values a and b, then a function f satisfies the approximation formula

$$f(x, y) \simeq f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

$$= p_1(x, y), \qquad \text{the Taylor polynomial of first order at } (a, b).$$

Do not spend too much time on this paragraph. The television programme contains a computer animation illustrating the fact that the two tangent lines to the section functions determine the tangent plane.

So when x and y are near a and b the surface with equation $z = p_1(x, y)$ is very close to the surface with equation $z = f(x, y)$, the surface associated with f. In fact, $z = p_1(x, y)$ is the equation of the tangent plane to this surface at $(a, b, f(a, b))$ as we shall now show.

First we show that the surface with equation $z = p_1(x, y)$ is indeed a plane. The equation $z = p_1(x, y)$, that is,

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \tag{14}$$

can be rearranged into the form

$$x\frac{\partial f}{\partial x}(a, b) + y\frac{\partial f}{\partial y}(a, b) - z$$

$$= a\frac{\partial f}{\partial x}(a, b) + b\frac{\partial f}{\partial y}(a, b) - f(a, b).$$

It is therefore the equation of a plane. To show it is the plane we are looking for, that is, the tangent plane, we show that its intersections with the section planes $x = a$ and $y = b$, which are straight lines, are the tangent lines to the surface $f(x, y)$ lying in those section planes. Consider first the section plane $y = b$ (see Figure 3).

The general equation of a plane is given in Section 3 of *Unit 14*.

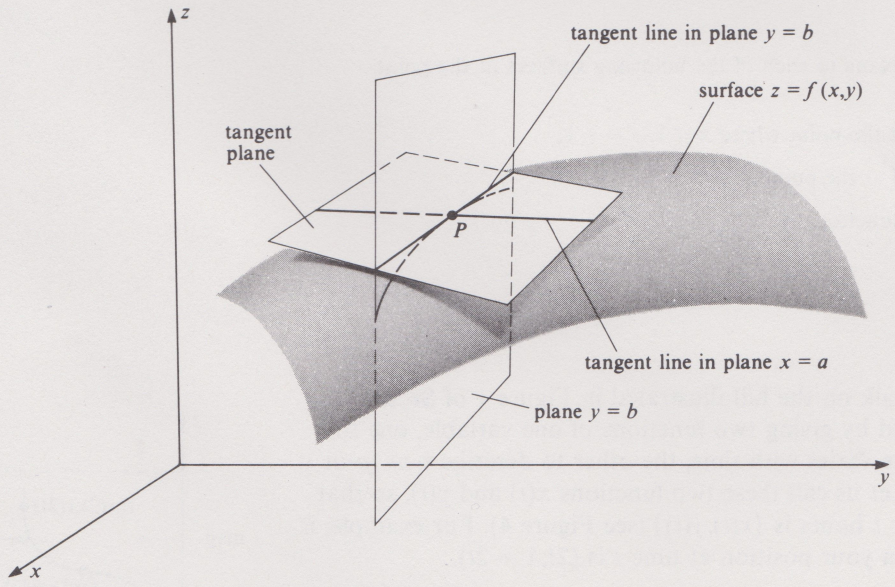


Figure 3

The intersections of the plane $z = p_1(x, y)$ and of the surface $z = f(x, y)$ with this section plane are described by the section functions $p_1(x, b)$ and $f(x, b)$. To show

that the former intersection is the tangent line of the latter it is sufficient to show that they both pass through the point $(a, b, f(a, b))$ and have the same slope there. So we want to show that the section function $p_1(x, b)$ and $f(x, b)$ have the same value at $x = a$ and also the same derivatives. Using (14) we see that

$$p_1(a, b) = f(a, b)$$

and $$\frac{d}{dx}p_1(x, b) = \frac{\partial f}{\partial x}(a, b)$$

and so the intersection of the proposed tangent plane with the section plane $y = b$ does have the right tangency property. In the same way we can check the corresponding property in the section plane $x = a$, and so complete the demonstration that $z = p_1(x, y)$ is the tangent plane to $z = f(x, y)$ at the point where $x = a$ and $y = b$.

Tangent plane equation

The equation of the tangent plane at the point $(a, b, f(a, b))$ to the surface $z = f(x, y)$ is given by

$$z = p_1(x, y)$$

$$= f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \qquad (15)$$

where p_1 is the Taylor polynomial of first order for f at (a, b).

Example 4

Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point on the surface where $x = 1$, $y = 2$.

Solution

Let us put $f(x, y) = x^2 + y^2$. Then,

$$\frac{\partial f}{\partial x}(x, y) = 2x, \quad \frac{\partial f}{\partial y}(x, y) = 2y, \quad \text{and} \quad f(1, 2) = 5.$$

So by (15) the tangent plane at $(1, 2, 5)$ has equation

$$z = 5 + 2(x - 1) + 4(y - 2), \text{ or}$$

$$2x + 4y - z = 5.$$

Exercise 6

Find the equation of the tangent plane to each of the following surfaces at the point indicated:

(i) the surface $z = xe^{x+y} - xy$ at the point where $x = 1$, $y = -1$;

(ii) the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, 2, 3)$.

(Hint for (ii): first find a suitable function f.)

[*Solution on p. 40*]

2.4 The chain rule

Let us suppose you go for a walk on the hill illustrated in Figure 1 of Section 1. Your walk could be represented by giving two functions of one variable, one to describe how your x coordinate varies with time, the other to describe how your y coordinates varies with time. Let us call these two functions $x(t)$ and $y(t)$, so that your position on the map after t hours is $(x(t), y(t))$ (see Figure 4). For example, if $x(t) = 2t$ and $y(t) = 1 - 2t$ then your position at time t is $(2t, 1 - 2t)$.

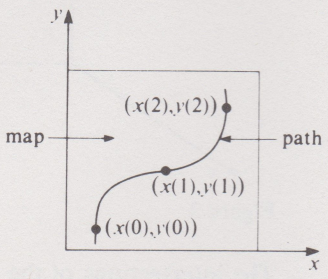


Figure 4

Besides your map coordinates, you may also want to know how your height varies with time. If the function of two variables representing the configuration of the land is $h(x, y)$, then your height at time t can be found by substituting the formulae $x(t)$ and $y(t)$ into this function. This gives a new formula giving the

height h in terms of the time, namely $h(x(t), y(t))$, which can be written $h(t)$ for short. For example, if the hill were a paraboloid (see Figure 5) with equation

$$h(x, y) = 1 - x^2 - y^2$$

(with x, y measured in kilometres) and your path were the one described above, then your height as a function of time is

$$h(t) = 1 - (2t)^2 - (1 - 2t)^2 = 4t - 8t^2.$$

As a walker you will be interested in the rate at which you are climbing or descending; this rate is the derivative $\dfrac{dh}{dt}$. This derivative will be related to how fast you walk and also to the topography of the land.

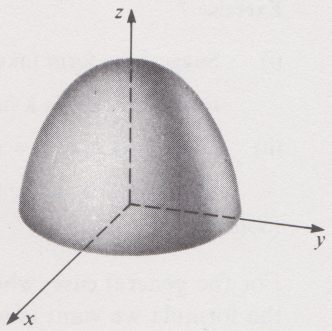


Figure 5

The purpose of this subsection is to introduce you to a formula which expresses this derivative quantitatively. It is a generalization of the chain rule for functions of one variable, and will be useful in the vector calculus unit which follows this one.

Let us first recall the chain rule for one-variable functions. In the form most useful to us here it says that if g is a function of x, and x is a function of t, then

$$\frac{dg}{dt} = \frac{dg}{dx}\frac{dx}{dt}. \tag{16}$$

For example, if $g(x) = \sin x$ and $x(t) = t^2$, so that $g(t) = \sin t^2$, then (16) gives

$$\frac{dg}{dt} = 2t\cos(t^2).$$

This chain rule gives a solution to the hill-walking problem, provided you walk along a suitable track. Specifically, suppose you walk parallel to the x-axis, say along the line $y = b$ where b is some constant. Then your height at time t is $h = h(x(t), b)$. To find how rapidly you are climbing or descending we differentiate with respect to t, obtaining

$$\frac{dh}{dt} = \frac{d}{dx}h(x(t), b)\frac{dx(t)}{dt}$$

or more concisely

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} \quad \text{if } y \text{ is constant.} \tag{17}$$

Notice the use of the partial derivative $\dfrac{\partial h}{\partial x}$ to indicate the derivative of $h(x, b)$, which is a section function of $h(x, y)$.

Example 5

If $h = 1 - x^2 - y^2$ as before, and your path is described by $x = t^2$, $y = \frac{1}{2}$, evaluate $\dfrac{dh}{dt}$, first directly and second by using the chain rule.

Solution

For the direct method we calculate h as a function of t by substitution:

$$h(t) = 1 - (t^2)^2 - (\tfrac{1}{2})^2 = \tfrac{3}{4} - t^4$$

and so $\dfrac{dh}{dt} = -4t^3$.

The chain rule method is

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} \qquad \text{(since } y \text{ is constant)}$$

$$= (-2x)(2t) = (-2t^2)(2t) = -4t^3 \qquad \text{(since } x = t^2\text{)},$$

the same result as the direct method.

Exercise 7

(i) Suggest the form taken by the chain rule for $\frac{dh}{dt}$ where h is a function of x and y, x is constant and y is a function of t.

(ii) Given that $h(x, y) = x^2y - xy^2$ and that $x = 4$ and $y = e^t$, evaluate $\frac{dh}{dt}$ first directly and second by using the chain rule.

[*Solution on p. 41*]

For the general case, where neither x nor y is constant, a convenient way to get the formula we want is to use the definition of a derivative:

$$\frac{dh}{dt} = \lim_{\delta t \to 0} \frac{h(t + \delta t) - h(t)}{\delta t}. \tag{18}$$

The most difficult term to deal with is $h(t + \delta t)$. First of all, its meaning, which is

$$h(t + \delta t) = h\big(x(t + \delta t), y(t + \delta t)\big). \tag{19}$$

You will not be expected to reproduce this proof.

Let us define, as in the previous subsection,

$$\begin{aligned} \delta x &= x(t + \delta t) - x(t); \\ \delta y &= y(t + \delta t) - y(t). \end{aligned} \tag{20}$$

Then the expression (19) can be written as

$$h(t + \delta t) = h(x(t) + \delta x, y(t) + \delta y).$$

Notice that h is both a function of the one variable t and of the two variables x and y.

By the first Taylor approximation this can be approximated as

$$h(t + \delta t) \simeq h(x, t) + \frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial y}\delta y \tag{21}$$

with an error that is small in comparison with δx and δy when both δx and δy are small.

Substituting (21) into (18) we find

$$\begin{aligned} \frac{dh}{dt} &= \lim_{\delta t \to 0} \frac{\frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial y}\delta y}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{\partial h}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial h}{\partial y}\frac{\delta y}{\delta t}. \end{aligned}$$

Using (20) and the definition of a derivative, this reduces to

$$\boxed{\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}.}$$

This is the **chain rule for functions of two variables**. Notice how it reduces to the simpler form given in (17) if y is constant, for then $\frac{dy}{dt} = 0$. There are analogous rules for functions of three or more variables, for example

$$\frac{dT}{dt}(x, y, z) = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}.$$

Example 6

In the hill-climbing example, we had $h = 1 - x^2 - y^2$, $x = 2t$, $y = 1 - 2t$.

Evaluate $\frac{dh}{dt}$ directly and also by using the chain rule.

Solution

Direct method: $h = 1 - x^2 - y^2 = 1 - (2t)^2 - (1 - 2t)^2 = 4t - 8t^2$

$$\frac{dh}{dt} = 4 - 16t.$$

Chain rule: $$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$
$$= (-2x)2 + (-2y)(-2)$$
$$= -4x + 4y$$
$$= -4(2t) + 4(1 - 2t) = 4 - 16t.$$

The results agree, as they should.

Exercise 8

If $h = x^2y - xy^2$, $x = t^2$, $y = t^3$ evaluate $\frac{dh}{dt}$ both directly and using the chain rule, and compare the answers.

[*Solution on p. 41*]

Summary of Section 2

1. The **Taylor polynomial of first order for a well-behaved function f at (a, b)** is
$$p_1(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$
2. For x and y near to a and b, $p_1(x, y)$ is approximately equal to $f(x, y)$; equivalently we have the **Taylor's approximation formulae**:
$$f(x, y) \simeq f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b), \text{ and}$$
$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$
3. *Errors in measurements* can be estimated using Taylor's approximation formulae and the **triangle inequality**
$$|a + b| \leq |a| + |b|.$$
4. For *functions of more than two variables* we also have first-order Taylor polynomials and Taylor's approximation formulae.
5. The **tangent plane** at $(a, b, f(a, b))$ to the surface $z = f(x, y)$ has equation
$$z = p_1(x, y).$$
6. The **chain rule** for functions of two variables is
$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$
where h is a function of x and y, x is a function of t, and y is a function of t.

End of section exercises

Exercise 9

Determine the first-order Taylor polynomials for the following functions at the indicated points:

(i) $T(x, y) = x^2y - 2xy^4 + x - y^2 - y$ at $(0, 0)$;

(ii) the same function at $(1, 1)$;

(iii) $f(x, y) = \frac{1}{2}(-1 + x^3 + y^2)^{3/2}$ at $(2, 3)$.

[*Solution on p. 41*]

Exercise 10

Using partial differentiation methods estimate the value of $\sqrt{1.97^3 + 0.99^6}$.

[*Solution on p. 41*]

Exercise 11

A person is determining the area T of a triangle by measuring two of its sides a, b and the included angle C (as shown in Figure 2). Using a ruler and protractor he decides that

$a = 0.5\,\text{m}$, $b = 0.4\,\text{m}$ and $C = \dfrac{\pi}{2}$, so that from the formula

$$T = \tfrac{1}{2}ab\sin C,$$

he decides that $T = 0.1\,\text{m}^2$. Unfortunately these instruments are not very accurate, and each of his measurements is subject to an error of up to 5%. Estimate the maximum error and the maximum percentage error in his determination of T.

[*Solution on p. 42*]

Exercise 12

Find the equation of the tangent plane to the surface $z = x^2 + xy - 2y^4$ at the point $(2, 1, 4)$.

[*Solution on p. 42*]

Exercise 13

Find the equation of the tangent plane to the surface

$$x^2 + xy + yz + zx = 1$$

at the point $(\frac{1}{2}, 0, \frac{3}{2})$.

[*Solution on p. 42*]

Exercise 14

Use the chain rule to evaluate $\dfrac{dh}{dt}$ when

$$h = x^2 + y^2, \qquad x = \cos t, \qquad y = \sin t.$$

Can you account for the result?

[*Solution on p. 42*]

3 Higher-order partial derivatives and Taylor polynomials

3.1 Higher-order partial derivatives

We have already seen how we can differentiate a function $f(x, y)$ of two variables x and y with respect to either x or y; this gave us the *first-order* partial derivatives of f, which we wrote (respectively) as $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

These two partial derivatives are themselves functions of x and y, and so may themselves be differentiable with respect to both the variables x and y. In the rest of the unit we will consider only functions that do have this property.

The partial derivative of the function $\dfrac{\partial f}{\partial x}$ with respect to x we will denote by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} \tag{1}$$

and its partial derivative with respect to y by

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial y\,\partial x}. \tag{2}$$

Similarly the partial derivative of the function $\dfrac{\partial f}{\partial y}$ with respect to x we will denote by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial x\,\partial y} \tag{3}$$

and its partial derivative with respect to y by

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2}. \tag{4}$$

We call the four functions (1), (2), (3) and (4) the **second-order partial derivatives** of f.

Example 1

Determine the second-order partial derivatives of the function

$$f(x,y) = e^x \cos y + x^2 - y + 1.$$

Solution

Differentiating f with respect to x and y successively, we get that

$$\frac{\partial f}{\partial x}(x,y) = e^x \cos y + 2x \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = -e^x \sin y - 1.$$

Differentiating $\dfrac{\partial f}{\partial x}$ with respect to x and y in turn, we get that

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y + 2 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y\,\partial x} = -e^x \sin y.$$

Similarly, differentiating $\dfrac{\partial f}{\partial y}$ with respect to x and y in turn, we get that

$$\frac{\partial^2 f}{\partial x\,\partial y} = -e^x \sin y \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = -e^x \cos y.$$

Exercise 1

Find the second-order partial derivatives of the following functions:

(i) $f(x,y) = \sqrt{xy}$ $\quad (x > 0,\ y > 0)$;

(ii) $f(x,y) = xe^{2x-y}$.

[*Solution on p. 42*]

The functions in (2) and (3) are called **mixed partial derivatives** since we are differentiating with respect to more than one of the variables involved. Notice that, in principle, we must be careful to differentiate the function f with respect to the variables in the correct order. Thus, for example $\dfrac{\partial^2 f}{\partial x \partial y}$ means 'differentiate f partially with respect to y and then differentiate the result partially with respect to x'. However we found that $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$ were equal in the above example. In fact this is not chance at all, in view of

The mixed derivative theorem

For any well-behaved function $f(x,y)$ of two independent variables x and y, the functions $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$ are equal.

If f is a function of more than two real variables, we can define second-order partial derivatives in exactly the same sort of way. Also, we can define third-order and higher-order partial derivatives in terms of partial derivatives of lower-order partial derivatives. For example, by $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial x}$ we mean $\dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial y\,\partial x}\right)$, or by $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial t}$ we mean $\dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial y\,\partial t}\right)$.

Exercise 2

Determine the second-order partial derivatives $\dfrac{\partial^2 f}{\partial x\,\partial y}$ and $\dfrac{\partial^2 f}{\partial z^2}$ of the following functions:

(i) $f(x,y,z) = e^{2x-z}\sin(3x+y)$;

(ii) $f(x,y,z) = x^2y^3z^4 + 2xy + 4z^2x^2 + y$.

[*Solution on p. 43*]

Exercise 3

Determine the partial derivatives $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial x}$, $\dfrac{\partial^3 f}{\partial y\,\partial x\,\partial x}$, $\dfrac{\partial^3 f}{\partial x\,\partial y\,\partial z}$ of the function $f(x, y, z) = x^3y^2z^4 + xy^3z^2$.

[*Solution on p. 43*]

3.2 Taylor polynomials of second order

We saw in Subsection 2.2 that

$$p_1(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b),$$

the Taylor polynomial of first order for f at (a, b), gives an approximation to the values of f near (a, b). In Section 4 we are going to need a polynomial which gives a better approximation to $f(x, y)$ near (a, b) than does $p_1(x, y)$. So we now determine such a polynomial $p_2(x, y)$ of the form

$$p_2(x, y) = p_2(a, b) + A(x - a) + B(y - a) + C(x - a)^2 + D(x - a)(y - b) + E(y - b)^2,$$

where A, B, C, D and E are constants. The approximation $f(x, y) \simeq p_2(x, y)$ would obviously be a generalization of the second-order Taylor's approximation (approximation (4) on page 12) for functions of one variable. In other words we will find some constants in the above expression for p_2 such that the difference $f(x, y) - p_2(x, y)$ is small in comparison with both $(x - a)^2$ and $(y - b)^2$ for x and y near to a and b. The method is analogous to the one we used in Subsection 2.2, but we shall carry out the calculations to a closer degree of approximation. We retain at each stage terms which affect the constants we are looking for but discard any terms which would lead in the long run only to quantities which are small in comparison with $(x - a)^2$ and $(y - b)^2$.

So, let us suppose that x and y are close to a and b. If, for the moment, we keep y fixed and apply the second-order one-variable Taylor's approximation to the section function $f(x, y)$, we see that

Do not spend long on the derivation: you will not be expected to reproduce it.

$$f(x, y) \simeq f(a, y) + \frac{\partial f}{\partial x}(a, y)(x - a) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a, y)(x - a)^2; \tag{5}$$

since f is a function of two variables we must remember that the derivatives will be partial derivatives rather than 'ordinary' derivatives.

We now examine each term on the right-hand side of (5), estimating each only to a degree of approximation sufficient for our purpose. For the first term, applying the second-order one-variable Taylor's approximation to the section function $f(a, y)$, where a is held fixed, we deduce that

$$f(a, y) \simeq f(a, b) + \frac{\partial f}{\partial y}(a, b)(y - b) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a, b)(y - b)^2. \tag{6}$$

For the next term in (5) we apply the first-order one-variable Taylor's approximation to the function $\dfrac{\partial f}{\partial x}(a, y)$; this gives

$$\frac{\partial f}{\partial x}(a, y) \simeq \frac{\partial f}{\partial x}(a, b) + \frac{\partial^2 f}{\partial y\,\partial x}(a, b)(y - b). \tag{7}$$

Finally, it is sufficient to estimate the last derivative in (5) as

$$\frac{\partial^2 f}{\partial x^2}(a, y) \simeq \frac{\partial^2 f}{\partial x^2}(a, b). \tag{8}$$

Putting the pieces of information contained in (6), (7) and (8) into our basic approximation equation (5), we see that:

For values of x and y near to a and b

$$f(x,y) \simeq f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial x\,\partial y}(a,b)(x-a)(y-b) + \tfrac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2. \quad (9)$$

We call the right-hand side of (9) the **Taylor polynomial of second order for f at (a,b)**, and denote it by p_2.

The error in this approximation is small in comparison with $(x-a)^2$ and $(y-b)^2$ when both $(x-a)$ and $(y-b)$ are sufficiently small.

Example 2

Determine the Taylor polynomial of second order for the function

$$f(x,y) = x^3 + xy - 2y^2 \qquad \text{at the point } (2,1).$$

Solution

Differentiating the formula for f with respect to x and y successively, we get

$$\frac{\partial f}{\partial x} = 3x^2 + y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x - 4y.$$

Differentiating once more, we deduce that

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = -4.$$

It follows that

$$f(2,1) = 8, \qquad \frac{\partial f}{\partial x}(2,1) = 13, \qquad \frac{\partial f}{\partial y}(2,1) = -2,$$

$$\frac{\partial^2 f}{\partial x^2}(2,1) = 12, \qquad \frac{\partial^2 f}{\partial x\,\partial y}(2,1) = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}(2,1) = -4.$$

Consequently, from the definition of p_2 in (9), we conclude that the Taylor polynomial of second order for f at $(2,1)$ is

$$p_2(x,y) = 8 + 13(x-2) - 2(y-1) + 6(x-2)^2 + (x-2)(y-1) - 2(y-1)^2.$$

Exercise 4

Determine the Taylor polynomial of second order for the same function at the point $(1,2)$.

[*Solution on p. 43*]

Exercise 5

Determine the Taylor polynomial of second order for each of the following functions at the indicated point:

(i) $f(x,y) = e^{x+y}$ at the point $(0,0)$;

(ii) $f(x,y) = e^{x+y}$ at the point $(1,2)$;

(iii) $f(x,y) = x\cos(x-y)$ at the point $(1,1)$.

[*Solution on p. 43*]

Summary of Section 3

1. By differentiating the partial derivative $\dfrac{\partial f}{\partial x}$ of a function $f(x,y)$ with respect to x and y we obtain the **second-order partial derivatives** $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$; and by

differentiating $\frac{\partial f}{\partial y}$ with respect to x and y the **second-order partial derivatives** $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$.

2. **The mixed derivative theorem**: For any well-behaved function f of two independent variables x and y, the functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are equal.

3. In a similar way we can define second-order partial derivatives for *functions of more than two variables*, and also *higher-order partial derivatives.*

4. The **Taylor polynomial of second order for $f(x, y)$ at (a, b)** is
$$p_2(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}(a, b)(x - a)^2$$
$$+ \frac{\partial^2 f}{\partial x \partial y}(a, b)(x - a)(y - b) + \tfrac{1}{2}\frac{\partial^2 f}{\partial y^2}(a, b)(y - b)^2.$$

5. For x and y near to a and b, $p_2(x, y)$ is **approximately equal** to $f(x, y)$. The approximation is at least as close as that given by $p_1(x, y)$ and in most cases is closer.

End of section exercises

Exercise 6

Determine the first- and second-order partial derivatives of the function
$$f(x, y) = e^x(1 + xy) + x^2 + xy - y^4.$$
[*Solution on p. 44*]

Exercise 7

Determine the derivatives $\frac{\partial^3 f}{\partial x^3}$ and $\frac{\partial^3 f}{\partial x \partial y \partial z}$ of the function $f(x, y, z) = xze^{x-y}$.
[*Solution on p. 44*]

Exercise 8

Determine the Taylor polynomials of second order for the function $f(x, y) = x^2y$ at the points $(1, -2)$ and $(-1, \frac{1}{2})$.
[*Solution on p. 44*]

4 Maxima and minima (Television and Tape Section)

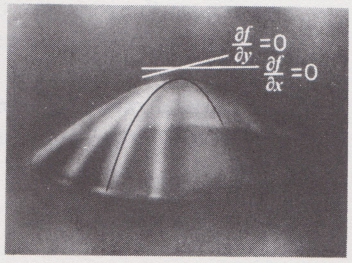
4.1 The stationary point criterion

One of the most straightforward applications of derivatives of functions of one variable is to the problem of determining when the given function has a maximum or a minimum. This is useful, for example, in curve sketching. Similarly one of the interesting and useful features of a function $f(x, y)$ of two variables is the location of its maxima and minima. In this section we first find a pair of simultaneous equations some of whose solutions are the maxima and minima we are looking for, and then find a way of characterizing the nature of (almost) all of their solutions.

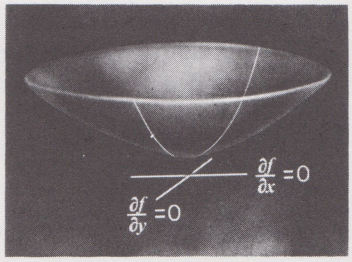
We say that a function f defined in a domain D in the x,y-plane has a **(local) maximum** at a point (a, b) in D if, for all points (x, y) in D with x and y sufficiently close to a and b,
$$f(x, y) \leq f(a, b).$$
Similarly we say that f has a **(local) minimum** at (a, b) if, for all points (x, y) in D with x and y sufficiently close to a and b,
$$f(x, y) \geq f(a, b).$$

For convenience in our discussion we will generally omit the qualification 'local', and we will use the term **extremum** to mean a point which is *either* a maximum *or* a minimum.

Extrema may occur either inside the domain D or on its boundary. An important case where extrema occur on the boundary is in linear programming (in that case D is the feasible region and f is the objective function). We saw in the unit on linear programming (*Unit 10*) that even though the function we were trying to optimize was linear in all its variables the extremum was quite hard to find, because of the complicated shape of the boundary. When the function is not linear, the problem of finding maxima and minima on the boundary is even more difficult and is beyond the scope of this course. On the other hand, the problem of finding maxima and minima which lie *inside* the domain D (that is, not on its boundary) can be solved quite simply in most cases, using partial differentiation. The rest of the section shows how to do this.



a maximum



a minimum

If a well-behaved function $f(x, y)$ has a maximum or a minimum at a point (a, b) inside D, then certainly its section function $f(x, b)$ (which we talked about in Subsection 1.3) has a maximum or a minimum (as x varies near a) at a. Its derivative with respect to x, namely $\dfrac{\partial f}{\partial x}(x, b)$, must be zero when $x = a$; that is, at the point (a, b). In the same way we find that $\dfrac{\partial f}{\partial y}(a, b)$ must also be zero if (a, b) is an extremum of f inside D. If we call a point (a, b) inside D where $\dfrac{\partial f}{\partial x}(a, b) = \dfrac{\partial f}{\partial y}(a, b) = 0$ a **stationary point** of f, we have thus established the

Stationary point criterion

If $f(x, y)$ has an extremum at the point (a, b) inside D then

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0.$$

Another (compact) way of putting this criterion is: *every extremum inside D is a stationary point.*

Before you watch the television programme, you should attempt the following exercise.

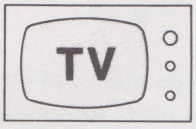
Exercise 1

Find the first-order partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ of the function

$$f(x, y) = xy(1 - x^2 - y^2).$$

(If you find this difficult see Subsection 1.3.)

[*Solution on p. 44*]



TV 25

Now view the television programme for Unit 25, 'Hunting the Hump'

The programme is about the function

$$f(x, y) = xy(1 - x^2 - y^2)$$

and the three-dimensional surface $z = f(x, y)$ associated with f. We notice first that the function f is zero on the x and y axes, and also on the circle $x^2 + y^2 = 1$ (with centre at the origin and radius one). By looking at the signs of the two factors xy and $1 - x^2 - y^2$ in the remaining portions of the plane, we see that the sign of f is as indicated in Figure 1.

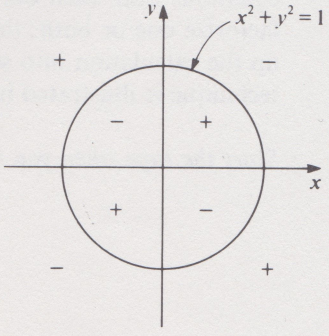
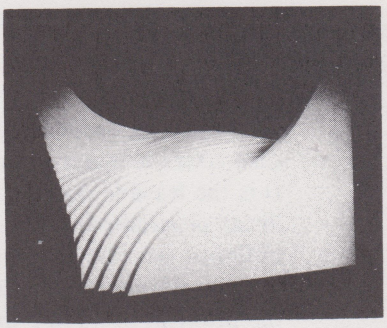


Figure 1. The sign of f

Since in the first quadrant of the circle $x^2 + y^2 < 1$ the function is positive, and on the boundary of this region the function is zero, it follows that f must have a maximum somewhere in the quadrant. Similarly f must have a maximum in the third quadrant of the circle, and a minimum in each of its second and fourth quadrants. To locate these maxima and minima we have to use the stationary point criterion; a simple-minded approach will not take us any further.

Using a computer animation, we see what we mean by the section functions of $f(x, y)$ and how we can build up the surface for f using them. The values of the partial derivatives of f at any point are just the slopes of the tangent lines of the corresponding section functions. The tangent plane to a surface at a point P is the plane through P that contains both these tangent lines.



The corrugations in the model are designed so that certain section functions are visible. Of course the actual surface would be smooth.

If f has a maximum at a point P the two section functions at P both have a maximum there; so the tangent lines at P are horizontal, and near P the surface for f lies below the tangent plane at P. Similarly if f has a minimum at P the surface has a horizontal tangent plane at P, and near P the surface for f lies above the tangent plane. By the stationary point criterion the maxima and minima of f occur at points (x, y) satisfying the equations

$$\frac{\partial f}{\partial x}(x, y) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x, y) = 0;$$

that is

$$y - 3x^2y - y^3 = 0 \qquad \text{and} \qquad x - x^3 - 3xy^2 = 0.$$

In the following tape commentary we solve this pair of simultaneous equations, and find that it has exactly 9 solutions:

(i) $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$;

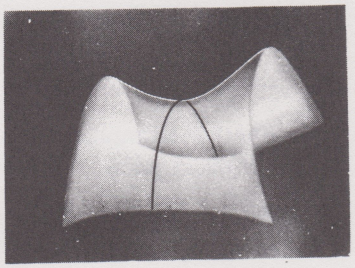
(ii) $(0, 0)$;

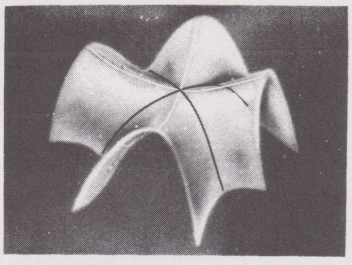
(iii) $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$.

Obviously the points in (i) are the four maxima and minima that we know must exist inside the four quadrants of the circle $x^2 + y^2 < 1$. Substituting the values of x and y into the formula $f(x, y) = xy(1 - x^2 - y^2)$ we see that

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = f(-\tfrac{1}{2}, -\tfrac{1}{2}) = \tfrac{1}{8} \qquad \text{and} \qquad f(\tfrac{1}{2}, -\tfrac{1}{2}) = f(-\tfrac{1}{2}, \tfrac{1}{2}) = -\tfrac{1}{8};$$

so the (local) maxima of the surface of f are of height $\frac{1}{8}$ above the x,y-plane and the (local) minima are a depth $\frac{1}{8}$ below the x,y-plane. By looking at the sign of f near the other five points we see that they are neither maxima nor minima; f is zero at each of the points, but yet it takes both positive and negative values in any small neighbourhood of each. This leads us to define a **saddle-point** of f to be a stationary point which is neither a maximum nor a minimum.





saddle-points

A method for simultaneous non-linear equations

We saw in the television programme that when we apply the stationary point criterion to find stationary points, we have to solve a pair of simultaneous equations:

$$\frac{\partial f}{\partial x}(x, y) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x, y) = 0.$$

As the equations are not usually linear we have to be rather careful in solving them, in order to both find *all* the possible solutions, and *not* discover values of x and y that are not actually solutions.

While there is no standard method analogous to Gaussian elimination for solving non-linear simultaneous equations, we generally proceed to add or subtract the equations from each other to get a more amenable pair in the sense that we can factorize one or both; then, by looking at where each factor vanishes, and splitting up the calculation into several possible cases, we can solve our equations. This technique is illustrated in the following tape.

Start the tape when you are ready.

This is the first of two tape sessions for this unit; listen only to the first part of the tape now.

1 A pair of Simultaneous Equations

$$y - 3x^2y - y^3 = 0$$
$$x - x^3 - 3y^2x = 0 \quad \Big\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

Factorize →

	$= 0$
	$= 0$

$\Big\} \ldots\ldots\ldots\ldots$ (B)

2 A Grid Scheme for Solution

	$x = 0$	$1 - x^2 - 3y^2 = 0$
$y = 0$	$(0, 0)$	$1 - x^2 = 0$ $(\ ,\ )(\ ,\ )$
$1 - 3x^2 - y^2 = 0$	$(\ ,\ )(\ ,\ )$	We look at this case in frame 3 →

3 A Simpler set of Equations

$$1 - 3x^2 - y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(C)}$$
$$1 - x^2 - 3y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(D)}$$

3(C) − (D) → $\quad = 0$

$x^2 =$

$y^2 =$ ← Substitute for x^2 in (D)

Solutions $\quad (\ ,\ ) \quad (\ ,\ ) \quad (\ ,\ ) \quad (\ ,\ )$

Consider all combinations to get four solutions

Exercise 2

Find the stationary points of the functions

(i) $f(x,y) = x^3 + y^3 - 3(x + y)$;

(ii) $f(x,y) = x^4 + \frac{1}{2}y^4 - x^2 - y^2 - x^2y^2$.

[*Solution on p. 44*]

4.2 The nature of stationary points

We saw that a stationary point of a function f of two variables may be a maximum of f or a minimum of f or (possibly) neither of these (that is, a saddle-point). To tell which it might be, we now look at the behaviour of f near a stationary point, (a, b) say, using the approximation to f near the point (a, b) given by the Taylor polynomial of second order. Suppose that the function $f(x, y)$ has a stationary point at (a, b); that is, suppose that

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0. \tag{1}$$

Then, if δx and δy are small, the Taylor approximation formula (9) on page 25 tells us that

$$f(a + \delta x, b + \delta y) \simeq f(a, b) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a, b)(\delta x)^2 + \frac{\partial^2 f}{\partial x\,\partial y}(a, b)\delta x\delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a, b)(\delta y)^2 \tag{2}$$

since the first derivative terms vanish, in view of (1).

For brevity we introduce the notation

$$A = \frac{\partial^2 f}{\partial x^2}(a, b), \qquad B = \frac{\partial^2 f}{\partial x\,\partial y}(a, b) \qquad \text{and} \qquad C = \frac{\partial^2 f}{\partial y^2}(a, b).$$

Then we can rewrite (2) as

$$f(a + \delta x, b + \delta y) - f(a, b) \simeq \tfrac{1}{2}[A(\delta x)^2 + 2B\delta x\delta y + C(\delta y)^2].$$

It follows that the sign of $f(a + \delta x, b + \delta y) - f(a, b)$ is the same as the sign of

$$A(\delta x)^2 + 2B\delta x\delta y + C(\delta y)^2. \tag{3}$$

So, by studying the sign of the expression (3) we can deduce whether $f(a + \delta x, b + \delta y) - f(a, b)$ is always positive, or always negative, or neither of these for small δx and δy; in other words, whether f has a minimum or a maximum or maybe neither a minimum nor a maximum at the point (a, b). (The details of this are given in an optional Appendix at the end of this section.) What we find is:

$AC - B^2$ criterion

Let (a, b) be a stationary point of $f(x, y)$, and

$$A = \frac{\partial^2 f}{\partial x^2}(a, b), \qquad B = \frac{\partial^2 f}{\partial x\,\partial y}(a, b) \qquad \text{and} \qquad C = \frac{\partial^2 f}{\partial y^2}(a, b).$$

(i) If $AC - B^2 > 0$, (a, b) is $\begin{cases} \text{a maximum if } A < 0. \\ \text{a minimum if } A > 0. \end{cases}$

(ii) If $AC - B^2 < 0$, (a, b) is a saddle-point.

Notice that this criterion gives *no* information on the nature of a stationary point at which $AC - B^2 = 0$; the study of such points is beyond the scope of this course. (In fact the point may be a maximum or a minimum or neither.) The use of this criterion is illustrated in the next example.

Example 1

Determine the nature of the stationary points of the function $f(x, y) = x^3 + y^3 - 3(x + y)$. (We met this function in Exercise 2.)

Solution

Differentiating the formulae

$$\frac{\partial f}{\partial x} = 3x^2 - 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3$$

with respect to x and y, we get that

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x\,\partial y} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 6y.$$

We find the values of these three second derivatives at the stationary points $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$ and present the results of these evaluations in a table, as below.

stationary point (a, b)	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(1, 1)$	$6(>0)$	0	6	$36(>0)$	minimum
$(-1, -1)$	$-6(<0)$	0	-6	$36(>0)$	maximum
$(1, -1)$	6	0	-6	$-36(<0)$	saddle-point
$(-1, 1)$	-6	0	6	$-36(<0)$	saddle-point

Exercise 3

Determine the nature of the stationary points of the function

$$f(x, y) = xy(1 - x^2 - y^2).$$

(We met this function in the television programme.)

[*Solution on p. 45*]

Exercise 4

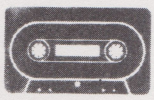
Find and classify the stationary points of the function

$$f(x, y) = \tfrac{4}{5}x^5 + y^5 - 2x^2 - 5y.$$

[*Solution on p. 46*]

Sometimes it takes a little care to solve the simultaneous equations whose solutions give the coordinates of the stationary points; in the next tape session we investigate one such case.

Start the tape when you are ready.



Listen to the last part of the tape here.

1 A Function of Two Variables

$$f(x, y) = x^2y - 2xy^2 - \frac{1}{6}x^3 + x + 2y + xy - \frac{1}{4}x^2 - y^2$$

$\frac{\partial f}{\partial x} =$
$\frac{\partial f}{\partial y} =$

2 Equations to Solve

Can we factorize?

$$2xy - 2y^2 - \frac{1}{2}x^2 + 1 + y - \tfrac{1}{2}x = 0 \quad \text{..... (A)}$$

$$-4xy + x^2 + 2 - 2y + x = 0 \quad \text{..... (B)}$$

2(A)+(B) → $\qquad = 0$ ..... (C)

$y = +1$	$y = -1$

Potential solutions

3 The strategy

	EQUATION (A)
$y = +1$	We look at this case in frame 4
$y = -1$	We look at this case in frame 5

4 The case with $y = +1$

$$2xy - 2y^2 - \tfrac{1}{2}x^2 + 1 + y - \tfrac{1}{2}x = 0 \quad \ldots\ldots\ldots\ldots \text{(A)}$$

Substitute $y = +1$ in (A)

	$= 0$

Factorize

	$= 0$

$x =$	$x =$

Solutions $(0, 1)$ and $(3, 1)$

5 The case with $y = -1$

$$2xy - 2y^2 - \tfrac{1}{2}x^2 + 1 + y - \tfrac{1}{2}x = 0 \quad \ldots\ldots\ldots\ldots \text{(A)}$$

Substitute $y = -1$ in (A)

	$= 0$

Solve using formula?

	$= 0$

$x =$	$x =$

Solutions

$(\quad , \quad)$	$(\quad , \quad)$

6 Second Partial Derivatives

Stationary Points

$(-4, -1)$ $(-1, -1)$ $(0, 1)$ $(3, 1)$

$$\frac{\partial f}{\partial x} = 2xy - 2y^2 - \tfrac{1}{2}x^2 + 1 + y - \tfrac{1}{2}x$$

$$\frac{\partial f}{\partial y} = x^2 - 4xy + 2 + x - 2y$$

$\dfrac{\partial^2 f}{\partial x^2} =$	$\dfrac{\partial^2 f}{\partial y^2} =$

$$\frac{\partial^2 f}{\partial x \partial y} =$$

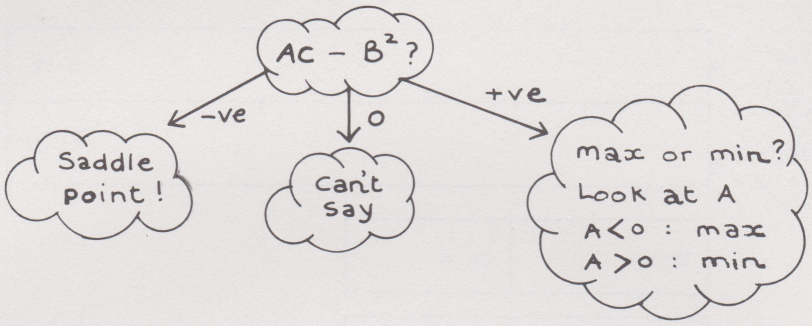
7 Classifying Stationary Points

$\frac{\partial^2 f}{\partial x^2} = 2y - x - \frac{1}{2}$ ← Call this A

$\frac{\partial^2 f}{\partial y^2} = -4x - 2$ ← Call this C

$\frac{\partial^2 f}{\partial x \partial y} = 2x - 4y + 1$ ← This MIXED derivative gets the MIDDLE letter B

Stationary Point	A	B	C	$AC - B^2$	classification
$(-4, -1)$					
$(-1, -1)$					
$(0, 1)$					
$(3, 1)$					



8 Solutions

Stationary Point	A	B	C	$AC - B^2$	classification
$(-4, -1)$	$\frac{3}{2}$ (>0)	-3	14	12 (>0)	Minimum
$(-1, -1)$	$-\frac{3}{2}$	3	2	-12 (<0)	saddle
$(0, 3)$	$\frac{3}{2}$	-3	-2	-12 (<0)	saddle
$(3, 1)$	$-\frac{3}{2}$ (<0)	3	-14	12 (>0)	Maximum

Exercise 5

Find and classify as far as you can the stationary points of the function

$$f(x, y) = x^2y - xy^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + xy + 3(x - y).$$

[*Solution on p. 46*]

Appendix (optional): proof of the $AC - B^2$ criterion

We saw earlier that, if (a, b) is a stationary point of the function $f(x, y)$, and if we put $A = \dfrac{\partial^2 f}{\partial x^2}(a, b)$, $B = \dfrac{\partial^2 f}{\partial x\,\partial y}(a, b)$ and $C = \dfrac{\partial^2 f}{\partial y^2}(a, b)$, then for all sufficiently small δx and δy we have

$$f(a + \delta x, b + \delta y) - f(a, b) \simeq \tfrac{1}{2}[A(\delta x)^2 + 2B\delta x\delta y + C(\delta y)^2] = \Delta, \text{ say.} \qquad (4)$$

Preliminary note

It is clear from (4) that, if $\Delta > 0$ for all small δx and δy (not both zero), then $f(a + \delta x, b + \delta y) > f(a, b)$ and so f has a minimum at (a, b); similarly, if $\Delta < 0$ for such δx and δy, then f has a maximum at (a, b). On the other hand, if we can always find a pair of arbitrarily small δx and δy (not both zero) such that $\Delta > 0$ and another pair of such δx and δy with $\Delta < 0$, then f has a saddle-point at (a, b). In this way information about the sign of Δ can tell us the nature of the stationary point.

Case 1: $AC - B^2 > 0$

Here $AC > B^2$ implies in particular that $AC > 0$ and so that $A \neq 0$. Therefore we can rewrite the right-hand side of (4) as:

$$\begin{aligned}\frac{2\Delta}{A} &= (\delta x)^2 + \frac{2B}{A}\delta x\delta y + \frac{C}{A}(\delta y)^2 \\ &= \left(\delta x + \frac{B}{A}\delta y\right)^2 + \left(\frac{C}{A} - \frac{B^2}{A^2}\right)(\delta y)^2 \qquad \text{(by completing the square)} \\ &= \left(\delta x + \frac{B}{A}\delta y\right)^2 + \frac{1}{A^2}(AC - B^2)(\delta y)^2. \qquad (5)\end{aligned}$$

This is the sum of two non-negative terms which cannot both vanish at the same time unless $\delta x = \delta y = 0$, which we do not allow. Hence $\dfrac{\Delta}{A} > 0$; and so Δ must have the same sign as A for all δx and δy not both zero.

It follows that if $A > 0$ we have a minimum, and if $A < 0$ we have a maximum of f.

Case 2: $AC - B^2 < 0$

(This always leads to a saddle-point, but the method of proof depends on whether $A = 0$ or not.) Suppose, first, that $A \neq 0$. We can complete the square just as in Case 1; but now since $AC - B^2 < 0$, the right-hand side of (5) can take either sign (being positive for $\delta x \neq 0$, $\delta y = 0$; and negative for $\delta y \neq 0$, $\delta x = -\dfrac{B}{A}\delta y$), so that f has a saddle-point. Next, suppose that $A = 0$. Then the formula (4) can be written as

$$\Delta = B\delta x\delta y + \tfrac{1}{2}C(\delta y)^2.$$

This can be made positive by choosing $\delta x = 0$, $\delta y \neq 0$. It can also be made negative by choosing some fixed negative δy and then choosing a much larger δx with the same sign as B (and we know that $B \neq 0$, since $AC - B^2 < 0$ and $A = 0$). So again we have a saddle-point.

Summary of Section 4

1. If, for all x and y sufficiently close to a and b, $f(x, y) \leq f(a, b)$, then f has a **(local) maximum** at the point (a, b); and if $f(x, y) \geq f(a, b)$ a **(local) minimum.**
2. An **extremum** is a stationary point which is either a maximum or a minimum.
3. A **stationary point** of a function $f(x, y)$ is a point (a, b) where $\dfrac{\partial f}{\partial x}(a, b) = \dfrac{\partial f}{\partial y}(a, b) = 0.$
4. **Stationary point criterion**: If a well-behaved function f has an extremum at (a, b), and (a, b) is not on the boundary of the domain of f, then $\dfrac{\partial f}{\partial x}(a, b) = 0$ and $\dfrac{\partial f}{\partial y}(a, b) = 0.$
5. A **saddle-point** of $f(x, y)$ is a stationary point which is neither a maximum nor a minimum.
6. A grid scheme can often be used to solve a pair of non-linear simultaneous equations. If neither equation can be factorized we may be able to take linear combinations of the equations to obtain one which does factorize.
7. **The $AC - B^2$ criterion:** Let (a, b) be a stationary point of $f(x, y)$; let $A = \dfrac{\partial^2 f}{\partial x^2}(a, b)$, $B = \dfrac{\partial^2 f}{\partial x \partial y}(a, b)$ and $C = \dfrac{\partial^2 f}{\partial y^2}(a, b)$.

   (i) If $AC - B^2 > 0$, (a, b) is $\begin{cases} \text{a maximum if } A < 0. \\ \text{a minimum if } A > 0. \end{cases}$

   (ii) If $AC - B^2 < 0$, (a, b) is a saddle-point.

End of section exercises

Exercise 6

Find and classify the stationary points of the function

$$f(x, y) = xy(6 - x - y).$$

[*Solution on p. 47*]

Exercise 7

Find and classify the stationary points of the function

$$f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$$

[*Solution on p. 47*]

5 End of unit exercises

Exercises on Section 1

Exercise 1

What are the section functions of the function $f(x, y) = \sin(x - y)$ at the point $\left(\dfrac{\pi}{2}, \pi\right)$?

Hence (or otherwise) determine the values of $\dfrac{\partial f}{\partial x}\left(\dfrac{\pi}{2}, \pi\right)$ and $\dfrac{\partial f}{\partial y}\left(\dfrac{\pi}{2}, \pi\right)$.

[*Solution on p. 48*]

Exercise 2

Determine the first-order partial derivatives of the following functions:

(i) $f(x, y) = x^3 - 3x^2y^4 + y^5$;

(ii) $g(x, t) = e^{xt^2} \sin(x - t)$;

(iii) $h(r, \theta) = e^{r \cos\theta} \sin\theta$;

(iv) $k(\rho, \theta, \phi) = \rho^2 \sin 2\theta \cos^2\phi$.

[*Solution on p. 48*]

Exercises on Section 2

Exercise 3

Determine the Taylor polynomial of first order for the following functions at the indicated points:

(i) $f(x, y) = x^5 - 2x^2y^3 + y^4 + 1$ at $(1, 1)$;

(ii) the same function at $(-1, -1)$;

(iii) $g(x, y) = e^{x/y}$ at $(2, 2)$.

[*Solution on p. 48*]

Exercise 4

Using an appropriate first-order Taylor polynomial, estimate the value of

$$\sqrt{5.97^2 + 4.02.^3}$$

[*Solution on p. 48*]

Exercise 5

A certain quantity f is calculated experimentally from measurements p, θ and R by the formula

$$f = \frac{p^2 \tan\theta}{R}.$$

In a particular experimental set-up, p, θ and R are measured to be 2m, $\frac{\pi}{4}$ radians, and 1m, giving a calculated value 4m for f. If the measurements of p, θ and R are subject to errors of 1%, 2% and 1% respectively, estimate (to the nearest percentage) the maximum relative error in the calculated value of f.

[*Solution on p. 48*]

Exercise 6

Determine the equation of the tangent plane at the point $(1, -1, -3)$ to the surface with equation

$$z = x^4 + 2x^2y - y^4.$$

[*Solution on p. 49*]

Exercise 7

Find the equation of the tangent plane at the point $(1, 1, 1)$ to the surface with equation

$$2e^{x+y} = e^{x+z} + e^{y+z}.$$

[*Solution on p. 49*]

Exercise 8

Use the chain rule to evaluate $\frac{df}{dt}$ where $f(u, v) = uv$, $u = e^t + e^{-t}$ and $v = e^t - e^{-t}$. Check your result by evaluating $\frac{df}{dt}$ directly.

[*Solution on p. 49*]

Exercises on Section 3

Exercise 9

Determine the second-order partial derivatives of the following functions:

(i) $f(x, y) = xy \sin(xy^2)$;

(ii) $g(r, \theta, t) = e^{-r^2t} \sin\theta$.

[*Solution on p. 49*]

Exercise 10

Show that the function $f(x, y) = xe^x \cos y$ satisfies the equation

$$\frac{\partial^4 f}{\partial x\,\partial x\,\partial x\,\partial x} + 2\frac{\partial^4 f}{\partial x\,\partial x\,\partial y\,\partial y} + \frac{\partial^4 f}{\partial y\,\partial y\,\partial y\,\partial y} = 0.$$

[*Solution on p. 49*]

Exercise 11

Determine the Taylor polynomial of second order for the function $f(x, y) = e^{1-x/y}$ at the point $(1, 1)$.

[*Solution on p. 50*]

Exercises on Section 4

Exercise 12

Find and classify the stationary points of the function

$$f(x, y) = xy - x - y - (x - 1)^2 - (y - 2)^2.$$

[*Solution on p. 50*]

Exercise 13

Find and classify the stationary points of the function

$$f(x, y) = \tfrac{1}{3}x^3 + y^3 + x^2 - 3(x + y).$$

[*Solution on p. 50*]

Appendix

Solutions to the exercises in Section 1

1. (i) Differentiating f partially with respect to x and y in turn gives

$$\frac{\partial f}{\partial x}(x, y) = \log y \frac{\partial}{\partial x}(x^2) = 2x \log y;$$

$$\frac{\partial f}{\partial y}(x, y) = x^2 \frac{\partial}{\partial y}(\log y) = \frac{x^2}{y}.$$

(ii) Differentiating g partially with respect to x and y gives

$$\frac{\partial g}{\partial x}(x, y) = 1 + 4xy;$$

$$\frac{\partial g}{\partial y}(x, y) = -2y + 2x^2.$$

(iii) Differentiating h partially with respect to x, y and t in turn gives

$$\frac{\partial h}{\partial x}(x, y, t) = e^t; \qquad \frac{\partial h}{\partial y}(x, y, t) = -2ye^{2t};$$

$$\frac{\partial h}{\partial t}(x, y, t) = xe^t - 2y^2e^{2t}.$$

Substituting $x = y = 1$ and $x = 4$, $y = 1$, $t = 0$ in the appropriate formulae in (i) and (iii), we deduce that

$$\frac{\partial f}{\partial y}(1, 1) = 1 \qquad \text{and} \qquad \frac{\partial h}{\partial t}(4, 1, 0) = 2.$$

2. (i) Differentiating f partially with respect to x and y in turn gives

$$\frac{\partial f}{\partial x}(x, y) = 1 - 3x^2 - 2y$$

and

$$\frac{\partial f}{\partial y}(x, y) = -2x + 2y.$$

(ii) Differentiating g partially with respect to x and y in turn gives

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial}{\partial x}(x)\sin(x - y) + x\frac{\partial}{\partial x}(\sin(x - y))$$

$$= \sin(x - y) + x\cos(x - y), \quad \text{and}$$

$$\frac{\partial g}{\partial y}(x, y) = x\frac{\partial}{\partial y}(\sin(x - y)) = -x\cos(x - y).$$

(iii) Differentiating h partially with respect to x, y and t in turn gives

$$\frac{\partial h}{\partial x}(x, y, t) = \frac{\partial}{\partial x}(e^{x-t})\cos(y + t) = e^{x-t}\cos(y + t),$$

$$\frac{\partial h}{\partial y}(x, y, t) = e^{x-t}\frac{\partial}{\partial y}(\cos(y + t)) = -e^{x-t}\sin(y + t) \quad \text{and}$$

$$\frac{\partial h}{\partial t}(x, y, t) = \frac{\partial}{\partial t}(e^{x-t})\cos(y + t) + e^{x-t}\frac{\partial}{\partial t}(\cos(y + t))$$

$$= -e^{x-t}\cos(y + t) - e^{x-t}\sin(y + t).$$

Substituting $x = 1$, $y = -1$ and $x = 0$, $y = \frac{\pi}{2}$, $t = 0$ in the appropriate formulae in (i) and (iii), we deduce that

$$\frac{\partial f}{\partial x}(1, -1) = 0 \qquad \text{and} \qquad \frac{\partial h}{\partial t}\left(0, \frac{\pi}{2}, 0\right) = -1.$$

3. (i) Here $\frac{\partial T}{\partial x} = 2xy - 2y^4$; $\qquad \frac{\partial T}{\partial y} = x^2 - 8xy^3$.

(ii) Here $\frac{\partial f}{\partial x} = -ye^{-xy}\sin x + e^{-xy}\cos x + 2x$;

$$\frac{\partial f}{\partial y} = -xe^{-xy}\sin x - 2y.$$

(iii) Here $\frac{\partial g}{\partial x} = \frac{t}{x^2}e^{-t/x}\sin(2t)$;

$$\frac{\partial g}{\partial t} = -\frac{1}{x}e^{-t/x}\sin(2t) + 2e^{-t/x}\cos(2t).$$

(iv) Here $\frac{\partial h}{\partial x} = z^2\sin\left(\frac{x}{y}\right) + \frac{xz^2}{y}\cos\left(\frac{x}{y}\right)$;

$$\frac{\partial h}{\partial y} = -\frac{x^2z^2}{y^2}\cos\left(\frac{x}{y}\right);$$

$$\frac{\partial h}{\partial z} = 2xz\sin\left(\frac{x}{y}\right).$$

(v) Here $\frac{\partial k}{\partial r} = \frac{-1}{(1 - 2r\cos\theta + r^2)^2}(-2\cos\theta + 2r)$

$$= \frac{2(\cos\theta - r)}{(1 - 2r\cos\theta + r^2)^2}$$

$$\frac{\partial k}{\partial \theta} = \frac{-1}{(1 - 2r\cos\theta + r^2)^2}(-2r)(-\sin\theta)$$

$$= \frac{-2r\sin\theta}{(1 - 2r\cos\theta + r^2)^2}.$$

Solutions to the exercises in Section 2

1. (i) Substituting the values $x = 4$, $y = 9$ in the formulae in Example 1, we get

$$f(4, 9) = 6, \quad \frac{\partial f}{\partial x}(4, 9) = \tfrac{1}{2}(4)^{-1/2}(9)^{1/2} = \tfrac{3}{4}$$

and

$$\frac{\partial f}{\partial y}(4, 9) = \tfrac{1}{2}(4)^{1/2}(9)^{-1/2} = \tfrac{1}{3};$$

hence the Taylor polynomial of first order for f at $(4, 9)$ is

$$p_1(x, y) = f(4, 9) + \frac{\partial f}{\partial x}(4, 9)(x - 4) + \frac{\partial f}{\partial y}(4, 9)(y - 9)$$

$$= 6 + \tfrac{3}{4}(x - 4) + \tfrac{1}{3}(y - 9).$$

Notice that the Taylor polynomials for f at different points are different; this is why we specified that p_1 is the Taylor polynomial for f *at* (a, b).

(ii) Differentiating f with respect to x and y, we get

$$\frac{\partial f}{\partial x}(x, y) = 5x^4 + 6xy - y^3$$

and

$$\frac{\partial f}{\partial y}(x, y) = 3x^2 - 3xy^2;$$

so that

$$f(1, 2) = -1, \qquad \frac{\partial f}{\partial x}(1, 2) = 9$$

and

$$\frac{\partial f}{\partial y}(1,2) = -9.$$

Hence the Taylor polynomial of first order for f at $(1,2)$ is

$$p_1(x,y) = -1 + 9(x-1) - 9(y-2).$$

2. Here we shall determine the Taylor polynomial p_1 of first order for the function $f(x,y) = \sqrt{x^2+y^2}$ at the point $(3,4)$. Then $p_1(2.98, 4.01)$ will be an estimate of the required number. Differentiating f partially, we see that

$$\frac{\partial f}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

so that $f(3,4) = 5$, $\dfrac{\partial f}{\partial x}(3,4) = \frac{3}{5}$ and $\dfrac{\partial f}{\partial y}(3,4) = \frac{4}{5}$.

Hence the Taylor polynomial of first order for f at $(3,4)$ is

$$p_1(x,y) = 5 + \tfrac{3}{5}(x-3) + \tfrac{4}{5}(y-4).$$

Then an estimate for $f(2.98, 4.01)$ is

$$p_1(2.98, 4.01) = 5 + \tfrac{3}{5}(-0.02) + \tfrac{4}{5}(0.01) = 4.996.$$

(In fact using a calculator gives $f(2.98, 4.01) = 4.9960484\ldots$.)

3. We saw in Example 1 that, for the function $m(x,y) = \sqrt{xy}$,

$$\frac{\partial m}{\partial x} = \tfrac{1}{2}\sqrt{y/x} \quad \text{and} \quad \frac{\partial m}{\partial y} = \tfrac{1}{2}\sqrt{x/y}.$$

We know that $m(4,9) = 6$ and the quantity that we are trying to estimate is $m(3.96, 8.91)$; so if p_1 is the Taylor polynomial of first order for m at $(4,9)$, a reasonable estimate will be $p_1(3.96, 8.91)$. The formulae for m and its derivatives give that

$$m(4,9) = 6, \qquad \frac{\partial m}{\partial x}(4,9) = \tfrac{3}{4}$$

and

$$\frac{\partial m}{\partial y}(4,9) = \tfrac{1}{3};$$

hence the Taylor polynomial p_1 is given by

$$p_1(x,y) = 6 + \tfrac{3}{4}(x-4) + \tfrac{1}{3}(y-9).$$

The desired numerical estimate is thus

$$p_1(3.96, 8.91) = 6 + \tfrac{3}{4}(-0.04) + \tfrac{1}{3}(-0.09) = 5.94.$$

(In fact $m(3.96, 8.91) = 5.94$ exactly!)

4. Now $c = \sqrt{a^2 + b^2 - 2ab\cos C}$; hence, using the notation of (10) on page 15,

$$\begin{aligned}\delta c &\simeq \frac{\partial c}{\partial a}\delta a + \frac{\partial c}{\partial b}\delta b \\ &= \frac{a - b\cos C}{\sqrt{a^2+b^2-2ab\cos C}}\delta a \\ &\qquad + \frac{b - a\cos C}{\sqrt{a^2+b^2-2ab\cos C}}\delta b \\ &= \frac{a - b\cos C}{c}\delta a + \frac{b - a\cos C}{c}\delta b.\end{aligned}$$

We are told that $|\delta a| \le 0.1$ and $|\delta b| \le 0.1$; so, if we apply the triangle inequality, and remember that we are considering c as a function of a and b near the point $(160, 210)$, with the constant $C = \pi/3$, we get

$$\begin{aligned}|\delta c| &\simeq \left|\frac{a - b\cos C}{c}\delta a + \frac{b - a\cos C}{c}\delta b\right| \\ &\le \left|\frac{a - b\cos C}{c}\delta a\right| + \left|\frac{b - a\cos C}{c}\delta b\right| \\ &\le \frac{55}{190}(0.1) + \frac{130}{190}(0.1) = 0.09736\ldots \text{m}.\end{aligned}$$

This is an estimate of the maximum error in the third side.

5. The stated formula gives L as a function of the three variables x, y and z; we are interested in the length, $3 + \delta L$, which corresponds to values $1 + \delta x$, $2 + \delta y$ and $2 + \delta z$ of x, y and z. This is given by (the extended version of) Taylor's approximation:

$$\begin{aligned}\delta L &\simeq \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial z}\delta z \\ &= (x^2+y^2+z^2)^{-1/2}(x\delta x + y\delta y + z\delta z).\end{aligned}$$

which we may also write more conveniently, as

$$\delta L \simeq \frac{x}{L}\delta x + \frac{y}{L}\delta y + \frac{z}{L}\delta z.$$

We are told that $|\delta x| \le 0.03$, $|\delta y| \le 0.03$ and $|\delta z| \le 0.03$; so, if we apply the triangle inequality (generalized in the obvious way to the case of three terms) and remember that we are considering L as a function of x, y and z near the point $(1,2,2)$, we get

$$\begin{aligned}|\delta L| &\simeq \left|\frac{x}{L}\delta x + \frac{y}{L}\delta y + \frac{z}{L}\delta z\right| \\ &\le \frac{x}{L}|\delta x| + \frac{y}{L}|\delta y| + \frac{z}{L}|\delta z| \\ &\le \tfrac{1}{3}(0.03) + \tfrac{2}{3}(0.03) + \tfrac{2}{3}(0.03) = 0.05\,m.\end{aligned}$$

This is an estimate of the maximum error in the length of the diagonal.

6. (i) Let us put $f(x,y) = xe^{x+y} - xy$, which has partial derivatives

$$\frac{\partial f}{\partial x} = e^{x+y} + xe^{x+y} - y \quad \text{and}$$

$$\frac{\partial f}{\partial y} = xe^{x+y} - x;$$

hence

$$f(1,-1) = 1 + 1 = 2,$$

$$\frac{\partial f}{\partial x}(1,-1) = 1 + 1 + 1 = 3$$

and

$$\frac{\partial f}{\partial y}(1,-1) = 1 - 1 = 0.$$

It follows that the tangent plane to the surface at the point $(1,-1,2)$ is

$$\begin{aligned}z &= f(1,-1) + \frac{\partial f}{\partial x}(1,-1)(x-1) + \frac{\partial f}{\partial y}(1,-1)(y-(-1)) \\ &= 2 + 3(x-1); \text{ that is} \\ z &= 3x - 1.\end{aligned}$$

(ii) Notice that the surface is a sphere with centre $(0,0,0)$. Here the equation of the surface is not in the desired form $z = f(x,y)$ for some function f. However, its equation can be rewritten as $z^2 = 14 - x^2 - y^2$, so that either

$$z = \sqrt{14 - x^2 - y^2} \quad \text{or} \quad z = -\sqrt{14 - x^2 - y^2}.$$

The first expression for z corresponds to points on the upper half of the sphere and the second expression for z to the lower half of the sphere. Since the sphere has centre $(0,0,0)$, the point $(1,2,3)$ clearly lies on the upper half of the sphere. So, near $(1,2,3)$ the surface has equation

$$z = f(x,y) = \sqrt{14 - x^2 - y^2}.$$

Hence

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{14 - x^2 - y^2}} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{14 - x^2 - y^2}},$$

so that

$$\frac{\partial f}{\partial x}(1,2) = -\tfrac{1}{3} \quad \text{and} \quad \frac{\partial f}{\partial y}(1,2) = -\tfrac{2}{3}.$$

Hence the tangent plane at $(1,2,3)$ to the surface has equation

$$z = 3 + (-\tfrac{1}{3})(x-1) + (-\tfrac{2}{3})(y-2);$$

i.e. $x + 2y + 3z = 14$.

7. (i) The problem is just the same as the one that led to (17) but with x and y interchanged; so the solution, obtained by interchanging x and y in (17), is

$$\frac{dh}{dt} = \frac{\partial h}{\partial y}\frac{dy}{dt} \text{ if } x \text{ is constant.}$$

(ii) Direct method: $h(x,y) = x^2y - xy^2$;

$h(4, e^t) = 16e^t - 4e^{2t}$;

so $\dfrac{dh}{dt} = 16e^t - 8e^{2t}$.

Chain rule: $h(x,y) = x^2y - xy^2$;

$$\frac{\partial h}{\partial y} = x^2 - 2xy; \qquad \frac{dy}{dt} = \frac{d}{dt}(e^t) = e^t;$$

so $$\frac{dh}{dt} = \frac{\partial h}{\partial y}\frac{dy}{dt} \qquad \text{(since } x \text{ is constant)}$$

$$= (x^2 - 2xy)e^t = (16 - 8e^t)e^t = 16e^t - 8e^{2t}.$$

This result is the same as for the direct method.

8. Direct method: $h = x^2y - xy^2 = t^4t^3 - t^2t^6 = t^7 - t^8$;

$$\frac{dh}{dt} = 7t^6 - 8t^7.$$

Chain rule:

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$

$$= (2xy - y^2)2t + (x^2 - 2xy)3t^2$$

$$= 2t(2t^5 - t^6) + 3t^2(t^4 - 2t^5)$$

$$= 4t^6 - 2t^7 + 3t^6 - 6t^7 = 7t^6 - 8t^7.$$

The two answers are the same, as they should be.

9. (i) Here

$$\frac{\partial T}{\partial x} = 2xy - 2y^4 + 1, \text{ and}$$

$$\frac{\partial T}{\partial y} = x^2 - 8xy^3 - 2y - 1.$$

Hence, since it follows that

$$\frac{\partial T}{\partial x}(0,0) = 1 \quad \text{and} \quad \frac{\partial T}{\partial y}(0,0) = -1,$$

the first-order Taylor polynomial for f at $(0,0)$ is

$$p_1(x,y) = T(0,0) + \frac{\partial T}{\partial x}(0,0)(x-0) + \frac{\partial T}{\partial y}(0,0)(y-0)$$

$$= x - y.$$

(ii) In this case

$$T(1,1) = -2, \qquad \frac{\partial T}{\partial x}(1,1) = 1 \qquad \text{and}$$

$$\frac{\partial T}{\partial y}(1,1) = -10,$$

so that the first-order Taylor polynomial for f at $(1,1)$ is

$$p_1(x,y) = T(1,1) + \frac{\partial T}{\partial x}(1,1)(x-1) + \frac{\partial T}{\partial y}(1,1)(y-1)$$

$$= -2 + (1)(x-1) + (-10)(y-1).$$

(iii) Differentiating $\frac{1}{2}(-1 + x^3 + y^2)^{3/2}$ with respect to x and y, we get that

$$\frac{\partial f}{\partial x} = \tfrac{3}{4}(-1 + x^3 + y^2)^{1/2}3x^2$$

$$= \tfrac{9}{4}(-1 + x^3 + y^2)^{1/2}x^2;$$

$$\frac{\partial f}{\partial y} = \tfrac{3}{4}(-1 + x^3 + y^2)^{1/2}2y$$

$$= \tfrac{3}{2}(-1 + x^3 + y^2)^{1/2}y;$$

hence

$$f(2,3) = 32, \qquad \frac{\partial f}{\partial x}(2,3) = 36 \qquad \text{and}$$

$$\frac{\partial f}{\partial y}(2,3) = 18.$$

Hence the first-order Taylor polynomial for f at $(2,3)$ is

$$p_1(x,y) = 32 + 36(x-2) + 18(y-3).$$

10. Here we shall look at the function $f(x,y) = \sqrt{x^3 + y^6}$ near the values $x = 2$, $y = 1$. If p_1 is the Taylor polynomial of first order for f at $(2,1)$, an estimate for the given number will be $p_1(1.97, 0.99)$. Differentiating f partially we get

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x^3 + y^6)^{-1/2}3x^2 \qquad \text{and}$$

$$\frac{\partial f}{\partial y} = \tfrac{1}{2}(x^3 + y^6)^{-1/2}6y^5;$$

so that

$$f(2,1) = 3, \qquad \frac{\partial f}{\partial x}(2,1) = 2 \qquad \text{and}$$

$$\frac{\partial f}{\partial y}(2,1) = 1.$$

Therefore the Taylor polynomial, p_1, is given by the formula

$$p_1(x, y) = 3 + 2(x - 2) + 1(y - 1).$$

The desired numerical estimate is thus

$$p_1(1.97, 0.99) = 2.93.$$

(Using a calculator gives $f(1.97, 0.99) = 2.930333\ldots$.)

11. If the exact values of the lengths of the sides and of the included angle are $a + \delta a$, $b + \delta b$ and $C + \delta C$, where $a = 0.5$, $b = 0.4$ and $C = \pi/2$, then we know that

$$\left|\frac{\delta a}{a}\right| \leq \frac{5}{100}, \quad \left|\frac{\delta b}{b}\right| \leq \frac{5}{100} \quad \text{and}$$

$$\left|\frac{\delta C}{C}\right| \leq \frac{5}{100};$$

in other words,

$$|\delta a| \leq \tfrac{1}{20}(0.5) = 0.025,$$

$$|\delta b| \leq \tfrac{1}{20}(0.4) = 0.02 \quad \text{and}$$

$$|\delta C| \leq \tfrac{1}{20}\left(\frac{\pi}{2}\right) = 0.0785398\ldots.$$

The three variables approximation formula on page 16 tells us that, if the exact value of the area is $T + \delta T$, where $T = 0.1\,\text{m}^2$, then

$$\delta T \simeq \frac{\partial T}{\partial a}\delta a + \frac{\partial T}{\partial b}\delta b + \frac{\partial T}{\partial C}\delta C$$

$$= \tfrac{1}{2}b\sin C\delta a + \tfrac{1}{2}a\sin C\delta b + \tfrac{1}{2}ab\cos C\delta C.$$

If we now apply the triangle inequality to this formula and substitute the values for a, b, C, we get that

$$|\delta T| = |\tfrac{1}{2}b\sin C\,\delta A + \tfrac{1}{2}a\sin C\,\delta b + \tfrac{1}{2}ab\cos C\,\delta C|$$

$$\leq 0.2|\delta a| + 0.25|\delta b| + 0.$$

Using the upper bounds for $|\delta a|$, $|\delta b|$ and $|\delta C|$ that we worked out above, it follows that an estimate for the upper bound of $|\delta T|$ is

$$(0.2)(0.025) + (0.25)(0.02) = 0.01.$$

The maximum percentage error in the determination of T is therefore approximately $\dfrac{0.01}{0.1} \times 100 = 10\,\%$.

12. Putting $f(x, y) = x^2 + xy - 2y^4$, we have that

$$\frac{\partial f}{\partial x}(x, y) = 2x + y \quad \text{and}$$

$$\frac{\partial f}{\partial y}(x, y) = x - 8y^3;$$

so that

$$\frac{\partial f}{\partial x}(2, 1) = 5 \quad \text{and}$$

$$\frac{\partial f}{\partial y}(2, 1) = -6.$$

Hence the equation of the tangent plane at (2, 1, 4) must be

$$z = f(2, 1) + \frac{\partial f}{\partial x}(2, 1)(x - 2) + \frac{\partial f}{\partial y}(2, 1)(y - 1)$$

$$= 4 + 5(x - 2) - 6(y - 1);$$

that is $5x - 6y - z = 0$.

13. We can rewrite the equation of the surface in the form

$$z = f(x, y) = \frac{1 - x^2 - xy}{x + y} = \frac{1}{x + y} - x.$$

Differentiating this with respect to x and y in turn, we have that

$$\frac{\partial f}{\partial x}(x, y) = \frac{-1}{(x + y)^2} - 1 \quad \text{and}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{-1}{(x + y)^2};$$

so that

$$\frac{\partial f}{\partial x}(\tfrac{1}{2}, 0) = -5 \quad \text{and} \quad \frac{\partial f}{\partial y}(\tfrac{1}{2}, 0) = -4.$$

Hence the equation of the tangent plane at the point $(\frac{1}{2}, 0, \frac{3}{2})$ must be

$$z = f(\tfrac{1}{2}, 0) + \frac{\partial f}{\partial x}(\tfrac{1}{2}, 0)(x - \tfrac{1}{2}) + \frac{\partial f}{\partial y}(\tfrac{1}{2}, 0)(y - 0)$$

$$= \tfrac{3}{2} + (-5)(x - \tfrac{1}{2}) + (-4)y$$

$$= 4 - 5x - 4y.$$

14. $$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$

$$= 2x(-\sin t) + 2y(\cos t)$$

$$= 2\cos t(-\sin t) + 2\sin t(\cos t) = 0.$$

The reason is that $x(t)^2 + y(t)^2 = \cos^2 t + \sin^2 t = 1$ so that $h = 1$ for all values of t and so $\dfrac{dh}{dt} = 0$.

Solutions to the exercises in Section 3

1. **(i)** We have already seen that

$$\frac{\partial f}{\partial x} = \frac{1}{2}\sqrt{y/x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{x/y}.$$

Differentiating each of these formulae first with respect to x and then with respect to y gives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -\tfrac{1}{4}x^{-3/2}y^{1/2},$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \tfrac{1}{4}x^{-1/2}y^{-1/2},$$

and

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \tfrac{1}{4}x^{-1/2}y^{-1/2},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -\tfrac{1}{4}x^{1/2}y^{-3/2}.$$

(ii) Differentiating f with respect to x and y in turn gives

$$\frac{\partial f}{\partial x} = e^{2x-y} + 2xe^{2x-y} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = -xe^{2x-y}.$$

Differentiating each of these formulae with respect to x and y gives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 4e^{2x-y} + 4xe^{2x-y},$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -e^{2x-y} - 2xe^{2x-y},$$

and

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -e^{2x-y} - 2xe^{2x-y},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = xe^{2x-y}.$$

2. **(i)** Differentiating f with respect to y and z gives

$$\frac{\partial f}{\partial y} = e^{2x-z}\cos(3x+y) \quad \text{and}$$

$$\frac{\partial f}{\partial z} = -e^{2x-z}\sin(3x+y).$$

Differentiating these formulae with respect to x and z respectively gives

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2e^{2x-z}\cos(3x+y) - 3e^{2x-z}\sin(3x+y)$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = e^{2x-z}\sin(3x+y).$$

(ii) Differentiating f with respect to y and z gives

$$\frac{\partial f}{\partial y} = 3x^2y^2z^4 + 2x + 1 \quad \text{and}$$

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + 8zx^2.$$

Differentiating these formulae with respect to x and z gives

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6xy^2z^4 + 2 \quad \text{and}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 12x^2y^3z^2 + 8x^2.$$

3. Differentiating f with respect to x and z gives

$$\frac{\partial f}{\partial x} = 3x^2y^2z^4 + y^3z^2 \quad \text{and}$$

$$\frac{\partial f}{\partial z} = 4x^3y^2z^3 + 2xy^3z.$$

We now differentiate $\dfrac{\partial f}{\partial x}$ with respect to y and x in turn; this gives

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 6x^2yz^4 + 3y^2z^2 \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6xy^2z^4.$$

Differentiating $\dfrac{\partial f}{\partial z}$ with respect to y we get

$$\frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 8x^3yz^3 + 6xy^2z.$$

If we now differentiate $\dfrac{\partial^2 f}{\partial y\,\partial x}$, $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y\,\partial z}$ with respect to x, y and x respectively, we conclude that

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial x} = 12xyz^4, \qquad \frac{\partial^3 f}{\partial y\,\partial x\,\partial x} = 12xyz^4,$$

and $$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z} = 24x^2yz^3 + 6y^2z.$$

4. Substituting $x = 1$, $y = 2$ in the formulae in Example 2, we get

$$f(1,2) = -5, \qquad \frac{\partial f}{\partial x}(1,2) = 5, \qquad \frac{\partial f}{\partial y}(1,2) = -7,$$

$$\frac{\partial^2 f}{\partial x^2}(1,2) = 6, \qquad \frac{\partial^2 f}{\partial x\,\partial y}(1,2) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(1,2) = -4.$$

Hence the Taylor polynomial of second order for f at $(1,2)$ is

$$p_2(x,y) = -5 + 5(x-1) - 7(y-2) + 3(x-1)^2 + (x-1)(y-2) - 2(y-2)^2.$$

5. **(i)** Differentiating f with respect to x and y successively gives

$$\frac{\partial f}{\partial x} = e^{x+y} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x+y}.$$

Differentiating once more, $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y^2} = e^{x+y}$.

Hence $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = e^0 = 1$ and

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = 1;$$

so that the Taylor polynomial of second order for f at $(0,0)$ is

$$p_2(x,y) = 1 + x + y + \tfrac{1}{2}x^2 + xy + \tfrac{1}{2}y^2.$$

(ii) Using the formulae for f and its derivatives from (i), we get

$$f(1,2) = \frac{\partial f}{\partial x}(1,2) = \frac{\partial f}{\partial y}(1,2) = e^3, \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x^2}(1,2) = \frac{\partial^2 f}{\partial x\,\partial y}(1,2) = \frac{\partial^2 f}{\partial y^2}(1,2) = e^3.$$

Hence the Taylor polynomial for f at $(1,2)$ is

$$p_2(x,y) = e^3 + e^3(x-1) + e^3(y-2) + \tfrac{1}{2}e^3(x-1)^2 + e^3(x-1)(y-2) + \tfrac{1}{2}e^3(y-2)^2.$$

(iii) Differentiating f twice with respect to x and y gives

$$\frac{\partial f}{\partial x} = \cos(x-y) - x\sin(x-y), \qquad \frac{\partial f}{\partial y} = x\sin(x-y),$$

$$\frac{\partial^2 f}{\partial x^2} = -2\sin(x-y) - x\cos(x-y),$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \sin(x-y) + x\cos(x-y),$$

and $$\frac{\partial^2 f}{\partial y^2} = -x\cos(x-y).$$

Hence

$$f(1,1) = 1, \qquad \frac{\partial f}{\partial x}(1,1) = 1, \qquad \frac{\partial f}{\partial y}(1,1) = 0,$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = -1, \qquad \frac{\partial^2 f}{\partial x\partial y}(1,1) = 1,$$

and $$\frac{\partial^2 f}{\partial y^2}(1,1) = -1;$$

so that the Taylor polynomial of second order for f at $(1,1)$ is

$$p_2(x,y) = 1 + 1(x-1) - \tfrac{1}{2}(x-1)^2 + (x-1)(y-1) - \tfrac{1}{2}(y-1)^2.$$

6. Differentiating f once with respect to each of x and y gives

$$\frac{\partial f}{\partial x} = e^x(1 + xy) + ye^x + 2x + y$$

$$= e^x(1 + xy + y) + 2x + y, \text{ and}$$

$$\frac{\partial f}{\partial y} = xe^x + x - 4y^3.$$

Differentiating again,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = e^x(1 + xy + y) + ye^x + 2$$

$$= e^x(1 + xy + 2y) + 2,$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = xe^x + e^x + 1, \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -12y^2.$$

In view of the mixed derivative theorem we do not need to calculate $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$, for it is the same as $\dfrac{\partial^2 f}{\partial x\,\partial y}$.

7. Differentiating f with respect to x and z gives

$$\frac{\partial f}{\partial x} = ze^{x-y} + xze^{x-y} \quad \text{and} \quad \frac{\partial f}{\partial z} = xe^{x-y};$$

differentiating these with respect to x and y respectively gives

$$\frac{\partial^2 f}{\partial x^2} = 2ze^{x-y} + xze^{x-y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y\,\partial z} = -xe^{x-y};$$

finally, differentiating these with respect to x gives that

$$\frac{\partial^3 f}{\partial x^3} = 3ze^{x-y} + xze^{x-y} \quad \text{and}$$

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z} = -e^{x-y} - xe^{x-y}.$$

8. Differentiating f with respect to x and y gives

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x\,\partial y} = 2x \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

Thus

$$f(1, -2) = -2, \quad \frac{\partial f}{\partial x}(1, -2) = -4, \quad \frac{\partial f}{\partial y}(1, -2) = 1,$$

$$\frac{\partial^2 f}{\partial x^2}(1, -2) = -4, \quad \frac{\partial^2 f}{\partial x\,\partial y}(1, -2) = 2 \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2}(1, -2) = 0;$$

so that the Taylor polynomial of second order for f at $(1, -2)$ is

$$p_2(x, y) = -2 - 4(x - 1) + (y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2).$$

Similarly,

$$f(-1, \tfrac{1}{2}) = \tfrac{1}{2} \quad \frac{\partial f}{\partial x}(-1, \tfrac{1}{2}) = -1, \quad \frac{\partial f}{\partial y}(-1, \tfrac{1}{2}) = 1,$$

$$\frac{\partial^2 f}{\partial x^2}(-1, \tfrac{1}{2}) = 1, \quad \frac{\partial^2 f}{\partial x\,\partial y}(-1, \tfrac{1}{2}) = -2 \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2}(-1, \tfrac{1}{2}) = 0;$$

so that the Taylor polynomial of second order for f at $(-1, \frac{1}{2})$ is

$$p_2(x, y) = \tfrac{1}{2} - 1(x + 1) + 1(y - \tfrac{1}{2}) + \tfrac{1}{2}(x + 1)^2 - 2(x + 1)(y - \tfrac{1}{2}).$$

Solutions to the exercises in Section 4

1. Differentiating $f(x, y) = xy - x^3y - xy^3$ partially with respect to x and y gives

$$\frac{\partial f}{\partial x} = y - 3x^2y - y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x - x^3 - 3xy^2.$$

2. **(i)** The first-order partial derivatives of f are

$$\frac{\partial f}{\partial x} = 3x^2 - 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3.$$

The coordinates (x, y) of the stationary points of f are thus determined by the two equations

$$3x^2 - 3 = 0 \quad \text{and} \quad 3y^2 - 3 = 0;$$

that is, $x^2 = 1$ and $y^2 = 1$.

These both have solutions ± 1. For either choice of x, either choice of y is possible. Hence the stationary points of f are

$$(1, 1), (-1, -1), (1, -1) \text{ and } (-1, 1).$$

(ii) The first-order partial derivatives of f are

$$\frac{\partial f}{\partial x} = 4x^3 - 2x - 2xy^2 \quad \text{and}$$

$$\frac{\partial f}{\partial y} = 2y^3 - 2y - 2x^2y;$$

so that the coordinates (x, y) of stationary points satisfy the simultaneous equations

$$0 = 4x^3 - 2x - 2xy^2 = 2x(2x^2 - 1 - y^2);$$

$$0 = 2y^3 - 2y - 2x^2y = 2y(y^2 - 1 - x^2). \qquad (1)$$

We now determine all the solutions of these using a table.

	$x = 0$	$2x^2 - 1 - y^2 = 0$
$y = 0$	Equations (1) satisfied. $(0,0)$ is a solution	Equations (1) become $2x(2x^2 - 1) = 0;$ $0 = 0.$ So $x = 0, \frac{1}{\sqrt{2}}$ or $\frac{-1}{\sqrt{2}}$ Hence the following are solutions of (1): $(0,0), \left(\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{-1}{\sqrt{2}}, 0\right)$.
$y^2 - 1 - x^2 = 0$	Equations (1) become $0 = 0;$ $2y(y^2 - 1) = 0.$ So $y = 0$, 1 or -1. Hence the following are solutions of (1): $(0,0)$, $(0,1)$ and $(0,-1)$.	Equations (1) become $0 = 0; 0 = 0.$ So all (x, y) satisfying the following equations are solutions of (1): $2x^2 - 1 - y^2 = 0$ $y^2 - 1 - x^2 = 0.$ These give $x^2 = 2$, $y^2 = 3$. Hence the following are solutions of (1): $(\sqrt{2}, \sqrt{3})$, $(-\sqrt{2}, -\sqrt{3})$, $(\sqrt{2}, -\sqrt{3})$, $(-\sqrt{2}, \sqrt{3})$.

Altogether we get nine distinct stationary points from this table.

3. From the solution to Exercise 1 we know the partial derivatives of f, namely:

$$\frac{\partial f}{\partial x} = y - 3x^2y - y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x - x^3 - 3xy^2.$$

Differentiating with respect to x and y, we get that

$$\frac{\partial^2 f}{\partial x^2} = -6xy, \quad \frac{\partial^2 f}{\partial x \partial y} = 1 - 3x^2 - 3y^2$$

and

$$\frac{\partial^2 f}{\partial y^2} = -6xy.$$

Then we use a table to look at each of the nine stationary points that we found in Subsection 4.1.

stationary point	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(\frac{1}{2}, \frac{1}{2})$	$-\frac{3}{2}$ (<0)	$-\frac{1}{2}$	$-\frac{3}{2}$	2 (>0)	maximum
$(-\frac{1}{2}, \frac{1}{2})$	$\frac{3}{2}$ (>0)	$-\frac{1}{2}$	$\frac{3}{2}$	2 (>0)	minimum
$(-\frac{1}{2}, -\frac{1}{2})$	$-\frac{3}{2}$ (<0)	$-\frac{1}{2}$	$-\frac{3}{2}$	2 (>0)	maximum
$(\frac{1}{2}, -\frac{1}{2})$	$\frac{3}{2}$ (>0)	$-\frac{1}{2}$	$\frac{3}{2}$	2 (>0)	minimum
(0, 0)	0	1	0	-1 (<0)	saddle-point
(1, 0)	0	-2	0	-4 (<0)	saddle-point
$(-1, 0)$	0	-2	0	-4 (<0)	saddle-point
(0, 1)	0	-2	0	-4 (<0)	saddle-point
(0, -1)	0	-2	0	-4 (<0)	saddle-point

The results of this classification using the $AC - B^2$ criterion are the same as those we discovered from geometric arguments in the television programme.

4. Differentiating f once,

$$\frac{\partial f}{\partial x} = 4x^4 - 4x \quad \text{and} \quad \frac{\partial f}{\partial y} = 5y^4 - 5.$$

Hence at stationary points,

$4x^4 - 4x = 0$ and $5y^4 - 5 = 0$;

that is, $4x(x^3 - 1) = 0$ and $5(y^4 - 1) = 0$.

So $x = 0$ or 1, and $y = +1$ or -1; and we check for the coordinates of stationary points using a grid as usual.

	$x = 0$	$x = 1$
$y = 1$	The point $(0, 1)$ is indeed a solution of the simultaneous equations	The point $(1, 1)$ is indeed a solution of the simultaneous equations
$y = -1$	The point $(0, -1)$ is indeed a solution of the simultaneous equations	The point $(1, -1)$ is indeed a solution of the simultaneous equations

So there are four stationary points $(0, 1)$, $(0, -1)$, $(1, 1)$ and $(1, -1)$.

Differentiating $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we see that

$$\frac{\partial^2 f}{\partial x^2} = 16x^3 - 4, \quad \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 20y^3.$$

We have now to look at the values of A, B, C and $AC - B^2$ at each stationary point. For clarity we lay out our results in tabular form as usual:

stationary point	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(0, 1)$	-4	0	20	-80 (<0)	saddle-point
$(0, -1)$	-4 (<0)	0	-20	80 (>0)	maximum
$(1, 1)$	12 (>0)	0	20	240 (>0)	minimum
$(1, -1)$	12	0	-20	-240 (<0)	saddle-point

5. Differentiating f once with respect to x and y we get that

$$\frac{\partial f}{\partial x} = 2xy - y^2 + x + y + 3 \quad \text{and}$$

$$\frac{\partial f}{\partial y} = x^2 - 2xy + x + y - 3.$$

Hence at stationary points we get the pair of equations

$$2xy - y^2 + x + y + 3 = 0; \tag{1}$$

$$x^2 - 2xy + x + y - 3 = 0. \tag{2}$$

There are several equally good ways in which we can solve this pair of non-linear simultaneous equations. For example, if we add (1) and (2) we get that

$$x^2 - y^2 + 2(x + y) = 0;$$

that is,

$$(x + y)(x - y + 2) = 0.$$

We now use a grid method to determine the set of solutions from this stage, in order not to 'lose' any solutions. We see that there are two cases to consider; namely $x + y = 0$ and $x - y + 2 = 0$.

which case?	stationary points?
$x + y = 0$ $(y = -x)$	Substituting for y in (1) gives that $0 = -2x^2 - x^2 + 3 = 3(1 - x^2)$, so that $x^2 = 1$ and hence $x = \pm 1$. This gives two stationary points: $(1, -1)$, $(-1, 1)$.
$x - y + 2 = 0$ $(y = x + 2)$	Substituting for y in (2) and rearranging terms gives $0 = x^2 + 2x + 1 = (x + 1)^2$, so that $x = -1$. This gives one stationary point $(-1, 1)$ (again!), making a total of two stationary points altogether.

Differentiating f a second time we find that

$$\frac{\partial^2 f}{\partial x^2} = 2y + 1, \quad \frac{\partial^2 f}{\partial y^2} = -2x + 1 \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x - 2y + 1.$$

We now look at the values of A, B, C and $AC - B^2$ at each stationary point, using the now familiar type of table:

stationary point	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(1, -1)$	-1	5	-1	$-24 (<0)$	saddle-point
$(-1, 1)$	3	-3	3	0	no information

(In fact, using a more detailed analysis of the behaviour of f near the point $(-1, 1)$ we can show that $(-1, 1)$ is also a saddle-point.)

6. Differentiating once, we get that

$$\frac{\partial f}{\partial x} = 6y - 2xy - y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 6x - x^2 - 2xy.$$

Hence at stationary points,

$$6y - 2xy - y^2 = 0 \quad \text{and} \quad 6x - x^2 - 2xy = 0;$$

that is,

$$y(6 - 2x - y) = 0 \quad \text{and} \quad x(6 - x - 2y) = 0, \qquad (1)$$

so either $y = 0$ or $6 - 2x - y = 0$; and either $x = 0$ or $6 - x - 2y = 0$.

	$y = 0$	$6 - 2x - y = 0$
$x = 0$	$(0, 0)$ is a solution of the pair (1)	Here $x = 0$ and $6 - y = 0$; so $(0, 6)$ is a solution of (1)
$6 - x - 2y = 0$	Here $y = 0$ and $6 - x = 0$; so $(6, 0)$ is a solution of (1)	Here $6 - 2x - y = 0$ and $6 - x - 2y = 0$; which give $(2, 2)$ as a solution of (1)

Differentiating f a second time, we see that

$$\frac{\partial^2 f}{\partial x^2} = -2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 6 - 2x - 2y \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2} = -2x.$$

We have now to look at the values of A, B, C and $AC - B^2$ at each of the four stationary points. Again we lay out conclusions in tabular form, for clarity:

stationary point	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(0, 0)$	0	6	0	$-36 (<0)$	saddle-point
$(2, 2)$	$-4 (<0)$	-2	-4	$12 (>0)$	maximum
$(0, 6)$	-12	-6	0	$-36 (<0)$	saddle-point
$(6, 0)$	0	-6	-12	$-36 (<0)$	saddle-point

7. Differentiating f once, we get that

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x \quad \text{and} \quad \frac{\partial f}{\partial y} = 6xy - 6y.$$

Hence at stationary points,

$$x^2 + y^2 - 2x = 0 \quad \text{and}$$

$$y(x - 1) = 0. \qquad (1)$$

	$x^2 + y^2 - 2x = 0$
$y = 0$	In this case the pair (1) become $x^2 - 2x = 0$ and $0 = 0$; and so have solutions $(0, 0)$ and $(2, 0)$.
$x - 1 = 0$	Here $x = 1$; so that the pair (1) become $y^2 - 1 = 0$ and $0 = 0$, and so have solutions $(1, 1)$ and $(1, -1)$.

Hence there are four stationary points $(0, 0)$, $(2, 0)$, $(1, 1)$ and $(1, -1)$.

Differentiating f a second time we see that

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6, \quad \frac{\partial^2 f}{\partial x \partial y} = 6y \quad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2} = 6x - 6.$$

We now look at the values of A, B, C and $AC - B^2$ at each stationary point, laying out our conclusions in a table.

stationary points	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(0,0)$	-6 (<0)	0	-6	36 (>0)	maximum
$(2,0)$	6 (>0)	0	6	36 (>0)	minimum
$(1,1)$	0	6	0	-36 (<0)	saddle-point
$(1,-1)$	0	-6	0	-36 (<0)	saddle-point

Solutions to the exercises in Section 5

1. The section functions at the point $\left(\frac{\pi}{2}, \pi\right)$ are

$$f\left(\frac{\pi}{2}, y\right) = \sin\left(\frac{\pi}{2} - y\right);$$

$$f(x, \pi) = \sin(x - \pi) = -\sin x.$$

Hence by the definition of partial derivatives,

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, \pi\right) = -\cos\left(\frac{\pi}{2}\right) = 0$$

$$\frac{\partial f}{\partial y}\left(\frac{\pi}{2}, \pi\right) = -\cos\left(-\frac{\pi}{2}\right) = 0.$$

2. **(i)** Differentiating f (partially) with respect to x and y successively, we get that

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy^4 \quad \text{and}$$

$$\frac{\partial f}{\partial y} = -12x^2y^3 + 5y^4.$$

(ii) Differentiating g (partially) with respect to x and t in turn, we get that

$$\frac{\partial g}{\partial x} = t^2 e^{xt^2} \sin(x - t) + e^{xt^2} \cos(x - t), \text{ and}$$

$$\frac{\partial g}{\partial t} = 2xte^{xt^2} \sin(x - t) - e^{xt^2} \cos(x - t).$$

(iii) Here,

$$\frac{\partial h}{\partial r} = e^{r\cos\theta} \sin\theta \cos\theta, \quad \text{and}$$

$$\frac{\partial h}{\partial \theta} = -e^{r\cos\theta} r \sin^2\theta + e^{r\cos\theta} \cos\theta.$$

(iv) Differentiating k (partially) with respect to ρ, θ and ϕ successively, we get that

$$\frac{\partial k}{\partial \rho} = 2\rho \sin 2\theta \cos^2\phi, \quad \frac{\partial k}{\partial \theta} = 2\rho^2 \cos 2\theta \cos^2\phi, \quad \text{and}$$

$$\frac{\partial k}{\partial \phi} = \rho^2 \sin 2\theta(-2 \sin\phi \cos\phi) = -\rho^2 \sin 2\theta \sin 2\phi.$$

3. **(i)** Differentiating f (partially) with respect to x and y, we get that

$$\frac{\partial f}{\partial x} = 5x^4 - 4xy^3 \quad \text{and}$$

$$\frac{\partial f}{\partial y} = -6x^2y^2 + 4y^3;$$

so that $\frac{\partial f}{\partial x}(1,1) = 1$ and $\frac{\partial f}{\partial y}(1,1) = -2$.

Also $f(1,1) = 1$. Hence the first-order Taylor polynomial for f at $(1,1)$ is

$$p_1(x,y) = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$$

$$= 1 + (x-1) - 2(y-1).$$

(ii) Here, $f(-1,-1) = 3$, $\frac{\partial f}{\partial x}(-1,-1) = 1$, and

$$\frac{\partial f}{\partial y}(-1,-1) = -10;$$

so that the first-order Taylor polynomial for f at $(-1,-1)$ is

$$p_1(x,y) = 3 + 1(x+1) - 10(y+1).$$

(iii) Differentiating (partially) with respect to x and y, we get that

$$\frac{\partial g}{\partial x} = \frac{1}{y} e^{x/y} \text{ and } \frac{\partial g}{\partial y} = \frac{-x}{y^2} e^{x/y}.$$

Hence $g(2,2) = e$, $\frac{\partial g}{\partial x}(2,2) = \frac{1}{2}e$ and

$$\frac{\partial g}{\partial y}(2,2) = -\frac{1}{2}e;$$

so that the first-order Taylor polynomial for g at $(2,2)$ is

$$p_1(x,y) = e + \tfrac{1}{2}e(x-2) - \tfrac{1}{2}e(y-2).$$

4. Here we shall determine the Taylor polynomial p_1 of first order for the function $f(x,y) = \sqrt{(x^2 + y^3)}$ at the point $(6,4)$. Then $p_1(5.97, 4.02)$ will be an estimate of the required number. Differentiating f partially, we get

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{(x^2 + y^3)}} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{2\sqrt{(x^2 + y^3)}}$$

so that $f(6,4) = 10$, $\frac{\partial f}{\partial x}(6,4) = \frac{3}{5}$ and

$$\frac{\partial f}{\partial y}(6,4) = \tfrac{12}{5}.$$

Hence the Taylor polynomial of first order for f at $(6,4)$ is

$$p_1(x,y) = 10 + \tfrac{3}{5}(x-6) + \tfrac{12}{5}(y-4);$$

so that the desired estimate is

$$p_1(5.97, 4.02) = 10 + \tfrac{3}{5}(-0.03) + \tfrac{12}{5}(0.02) = 10.03.$$

5. Here we use the three-variable Taylor approximation on page 16. f is given as a function of three variables p, R and θ by the formula

$$f = \frac{p^2 \tan\theta}{R}; \text{ hence,}$$

$$\delta f \simeq \frac{\partial f}{\partial p}\delta p + \frac{\partial f}{\partial \theta}\delta\theta + \frac{\partial f}{\partial R}\delta R$$

$$= \frac{2p\tan\theta}{R}\delta p + \frac{p^2\sec^2\theta}{R}\delta\theta - \frac{p^2\tan\theta}{R^2}\delta R. \qquad (1)$$

The value of f was calculated from the values $p = 2$, $\theta = \frac{\pi}{4}$ and $R = 1$; and the information about possible errors tells us that

$$\left|\frac{\delta p}{p}\right| \leq \frac{1}{100}, \qquad \left|\frac{\delta\theta}{\theta}\right| \leq \frac{2}{100} \qquad \text{and} \qquad \left|\frac{\delta R}{R}\right| \leq \frac{1}{100};$$

So if we apply the triangle inequality to (1) and substitute the values of p, θ and R we get an approximate maximum value for $|\delta f|$ by

$$|\delta f| \simeq \left|\frac{2p\tan\theta}{R}\delta p + \frac{p^2\sec^2\theta}{R}\delta\theta + \frac{p^2\tan\theta}{R^2}\delta R\right|$$

$$\leq 4|\delta p| + 8|\delta\theta| + 4|\delta R|.$$

Hence, since $|\delta p| \leq \frac{2}{100}$, $|\delta\theta| \leq \frac{\pi}{200}$ and $|\delta R| \leq \frac{1}{100}$, it follows that an approximate maximum value for $|\delta f|$ is

$$\frac{8}{100} + \frac{4\pi}{100} + \frac{4}{100} = \frac{12 + 4\pi}{100}.$$

The maximum percentage error in the calculated value of f is therefore approximately

$$\frac{|\delta f|}{f} \times 100 = \frac{12 + 4\pi}{4}\% \simeq 6\%$$

6. Putting $f(x, y) = x^4 + 2x^2y - y^4$, we have that

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2x^2 - 4y^3;$$

so that

$$f(1, -1) = -2, \qquad \frac{\partial f}{\partial x}(1, -1) = 0, \qquad \text{and}$$

$$\frac{\partial f}{\partial y}(1, -1) = 6.$$

Hence the equation of the tangent plane at $(1, -1)$ is

$$z = f(1, -1) + \frac{\partial f}{\partial x}(1, -1)(x - 1) + \frac{\partial f}{\partial y}(1, -1)(y + 1)$$

$$= -2 + 6(y + 1) = 4 + 6y.$$

7. We can rewrite the equation of the surface in the form $z = f(x, y)$, where

$$2e^{x+y} = (e^x + e^y)e^{f(x,y)}. \qquad (1)$$

Differentiating both sides of (1) partially with respect to x and y successively, we get that

$$2e^{x+y} = e^x e^{f(x,y)} + (e^x + e^y)e^{f(x,y)}\frac{\partial f}{\partial x}(x, y) \qquad (2)$$

and

$$2e^{x+y} = e^y e^{f(x,y)} + (e^x + e^y)e^{f(x,y)}\frac{\partial f}{\partial y}(x, y). \qquad (3)$$

Substituting $x = y = 1$ and using the fact $f(1, 1) = 1$ in (2) and (3) we get

$$\frac{\partial f}{\partial x}(1, 1) = \tfrac{1}{2} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(1, 1) = \tfrac{1}{2}.$$

Consequently the equation of the tangent plane at $(1, 1)$ to the surface is

$$z = 1 + \tfrac{1}{2}(x - 1) + \tfrac{1}{2}(y - 1) = \tfrac{1}{2}(x + y).$$

8. $$\frac{df}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

$$= v(e^t - e^{-t}) + u(e^t + e^{-t})$$

$$= (e^t - e^{-t})^2 + (e^t + e^{-t})^2 = 2e^{2t} + 2e^{-2t}.$$

Check: $f(t) = (e^t + e^{-t})(e^t - e^{-t}) = e^{2t} - e^{-2t}$, so

$$\frac{df}{dt} = 2e^{2t} + 2e^{-2t}, \text{ which checks.}$$

9. **(i)** Differentiating f partially with respect to x and y gives

$$\frac{\partial f}{\partial x} = y\sin(xy^2) + xy^3\cos(xy^2);$$

$$\frac{\partial f}{\partial y} = x\sin(xy^2) + 2x^2y^2\cos(xy^2).$$

Differentiating again gives

$$\frac{\partial^2 f}{\partial x^2} = 2y^3\cos(xy^2) - xy^5\sin(xy^2),$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \sin(xy^2) + 5xy^2\cos(xy^2) - 2x^2y^4\sin(xy^2), \text{ and}$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y\cos(xy^2) - 4x^3y^3\sin(xy^2).$$

Finally, by the mixed derivative theorem, $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial^2 f}{\partial x\,\partial y}$.

(ii) Differentiating g with respect to r, θ and t gives

$$\frac{\partial g}{\partial r} = -2rte^{-r^2t}\sin\theta, \quad \frac{\partial g}{\partial\theta} = e^{-r^2t}\cos\theta \text{ and}$$

$$\frac{\partial g}{\partial t} = -r^2e^{-r^2t}\sin\theta;$$

differentiating again gives

$$\frac{\partial^2 g}{\partial r^2} = -2te^{-r^2t}\sin\theta + 4r^2t^2e^{-r^2t}\sin\theta,$$

$$\frac{\partial^2 g}{\partial\theta^2} = -e^{-r^2t}\sin\theta, \quad \frac{\partial^2 g}{\partial r\,\partial\theta} = -2rte^{-r^2t}\cos\theta,$$

$$\frac{\partial^2 g}{\partial r\,\partial t} = -2re^{-r^2t}\sin\theta + 2r^3te^{-r^2t}\sin\theta,$$

$$\frac{\partial^2 g}{\partial\theta\,\partial t} = -r^2e^{-r^2t}\cos\theta \qquad \text{and}$$

$$\frac{\partial^2 g}{\partial t^2} = r^4e^{-r^2t}\sin\theta.$$

By the mixed derivative theorem these are all the distinct second-order derivatives.

10. Differentiating f partially with respect to x and y gives

$$\frac{\partial f}{\partial x} = e^x\cos y + xe^x\cos y \qquad \text{and}$$

$$\frac{\partial f}{\partial y} = -xe^x\sin y.$$

Differentiating the first equation with respect to x and the second with respect to y gives

$$\frac{\partial^2 f}{\partial x^2} = 2e^x \cos y + xe^x \cos y \qquad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2} = -xe^x \cos y.$$

Differentiating the first of this pair with respect to x and y in turn, and the second with respect to y, we get

$$\frac{\partial^3 f}{\partial x^3} = 3e^x \cos y + xe^x \cos y,$$

$$\frac{\partial^3 f}{\partial y\, \partial x^2} = -2e^x \sin y - xe^x \sin y,$$

and $$\frac{\partial^3 f}{\partial y^3} = xe^x \sin y.$$

Differentiating these with respect to x, y and y in turn, we obtain

$$\frac{\partial^4 f}{\partial x^4} = 4e^x \cos y + xe^x \cos y,$$

$$\frac{\partial^4 f}{\partial y^2\, \partial x^2} = -2e^x \cos y - xe^x \cos y,$$

and $$\frac{\partial^4 f}{\partial y^4} = xe^x \cos y.$$

It follows that

$$\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2\, \partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0$$

(using the mixed derivative theorem).

11. Differentiating f partially with respect to x and y in turn, we get that

$$\frac{\partial f}{\partial x} = -\frac{1}{y}e^{1-(x/y)} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x}{y^2}e^{1-(x/y)};$$

differentiating again, we conclude that

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{y^2}e^{1-(x/y)},$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{1}{y^2}e^{1-(x/y)} - \frac{x}{y^3}e^{1-(x/y)}, \text{ and}$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{2x}{y^3}e^{1-(x/y)} + \frac{x^2}{y^4}e^{1-(x/y)}.$$

At the point $(1, 1)$, we thus have

$$f(1,1) = e^0 = 1, \qquad \frac{\partial f}{\partial x}(1,1) = -1, \qquad \frac{\partial f}{\partial y}(1,1) = 1,$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 1, \qquad \frac{\partial^2 f}{\partial x\, \partial y}(1,1) = 0 \qquad \text{and}$$

$$\frac{\partial^2 f}{\partial y^2}(1,1) = -1.$$

So the second-order Taylor polynomial for f at $(1, 1)$ will be

$$\begin{aligned} p_2(x,y) &= f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(1,1)(x-1)^2 \\ &\quad + \frac{\partial^2 f}{\partial x\, \partial y}(1,1)(x-1)(y-1) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(1,1)(y-1)^2 \\ &= 1 - (x-1) + (y-1) + \tfrac{1}{2}(x-1)^2 - \tfrac{1}{2}(y-1)^2. \end{aligned}$$

12. Differentiating f with respect to x and y,

$$\frac{\partial f}{\partial x} = y - 1 - 2(x-1) \qquad \text{and}$$

$$\frac{\partial f}{\partial y} = x - 1 - 2(y-2).$$

Hence at a stationary point, where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$,

$$y - 2x + 1 = 0 \qquad \text{and} \qquad x - 2y + 3 = 0.$$

This pair of simultaneous equations has solution $x = \frac{5}{3}$, $y = \frac{7}{3}$; so there is just one stationary point $(\frac{5}{3}, \frac{7}{3})$.

Differentiating $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ with respect to x and y we find that

$$\frac{\partial^2 f}{\partial x^2} = -2, \qquad \frac{\partial^2 f}{\partial x\, \partial y} = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = -2.$$

Hence at $(\frac{5}{3}, \frac{7}{3})$, we have that

$$A = \frac{\partial^2 f}{\partial x^2} = -2, \text{ and}$$

$$AC - B^2 = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 3 > 0.$$

It follows that f has a maximum at $(\frac{5}{3}, \frac{7}{3})$.

13. Differentiating f with respect to x and y, we have that

$$\frac{\partial f}{\partial x} = x^2 + 2x - 3 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3y^2 - 3.$$

Hence at a stationary point, where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$,

$$0 = (x+3)(x-1) \qquad \text{and} \qquad 0 = 3(y-1)(y+1)$$

so that $x = -3$ or 1, and $y = 1$ or -1.

	$x = -3$	$x = 1$
$y = 1$	$(-3, 1)$ is a stationary point	$(1, 1)$ is a stationary point
$y = -1$	$(-3, -1)$ is a stationary point	$(1, -1)$ is a stationary point

It follows that there are four stationary points,

$(-3, 1)(-3, -1)$, $(1, 1)$ and $(1, -1)$.

Differentiating $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ partially with respect to x and y we get

$$\frac{\partial^2 f}{\partial x^2} = 2x + 2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2} = 6y.$$

We now investigate the nature of our stationary points using the usual table:

stationary point	A $\left(\frac{\partial^2 f}{\partial x^2}\right)$	B $\left(\frac{\partial^2 f}{\partial x \partial y}\right)$	C $\left(\frac{\partial^2 f}{\partial y^2}\right)$	$AC - B^2$	classification
$(-3, 1)$	-4	0	6	-24 (<0)	saddle-point
$(-3, -1)$	-4 (<0)	0	-6	24 (>0)	maximum
$(1, 1)$	4 (>0)	0	6	24 (>0)	minimum
$(1, -1)$	4	0	-6	-24 (<0)	saddle-point

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